APPLICATION OF THE RANDOM SET FINITE ELEMENT METHOD (RS-FEM) IN GEOTECHNICS

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INTRODUCTION

Dealing with uncertainty, caused e.g. by material parameters varying in a wide range or simply by a lack of knowledge, is one of the important issues in geotechnical analyses. The advantages of numerical modelling have been appreciated by practitioners, in particular when displacements and deformations of complex underground structures have to be predicted. Therefore, it seems to be logical to combine numerical modelling with concepts for the mathematical representation of uncertainties. Recent theoretical developments and advances made in computational modelling have established various methods which may serve as a basis for a more formal consideration of uncertainties as has been done so far. Random set theory offers one of these possibilities for the mathematical representation of uncertainties. It can be viewed as a generalisation of probability theory and interval analysis. After a brief introduction of the basics of the proposed approach an application to a boundary value problem is presented. The results show that the assessment of the probability of damage of a building, situated adjacent to the excavation, is in line with observed behaviour.

About half a century ago, the latest phase in a debate about the mathematisation of uncertainty started. Probability theory based on the neo-Bayesian school has been extensively employed (see e.g. Ang & Tang 1975, Wright & Ayton 1994) but more recently fuzzy methods and evidence theory have been established as legitimate extension of classical probability theory. It is important to realise that different sources of uncertainty exist, material parameters varying in a wide - but known - range are one of them but simply the lack of knowledge may actually be the more dominant one in geotechnical engineering. The full scope of uncertainty and its dual nature can be described by the definitions from Helton (1997): Aleatory uncertainty results from the fact that a parameter can behave in random ways whereas epistemic uncertainty results from the lack of knowledge about a parameter. Aleatory uncertainty is associated with variability in observable populations and is therefore not reducible, whereas epistemic uncertainty changes with ones' state of knowledge and is therefore reducible. Despite the fact that it is well recognised that the frequentist approach associated with classical probability theory is well suited for dealing with aleatory uncertainty, it is common practice to apply probability theory to characterise both types of uncertainty, although it is not capable of capturing epistemic uncertainty (Sentz & Ferson 2002).

In view of having insufficient data available for a particular project, in practice alternative sources of information are usually utilised. These sources can be previously published data for similar ground conditions, general correlations from literature or simply expert knowledge. This data conventionally appear as intervals with no information about the probability distribution across the interval and therefore it seems to be appropriate to actually work with these intervals rather than assuming a density distribution function. One way of doing so is the application of the random set theory as presented here.

RANDOM SET THEORY

The theory of random sets provides a general framework for dealing with set-based information and discrete probability distributions. The analysis gives the same result as interval analysis, when only range information is available and the same result as Monte-Carlo simulations when the information is abundant.

Basic concepts

Let X be a non-empty set containing all the possible values of a variable x. Dubois & Prade (1990, 1991) defined a random set on X as a pair $(\Im, m)$ where $\Im = \{A_i : i = 1,\ldots,n\}$ and m is a mapping, $\Im \rightarrow [0,1]$, so that $m(\emptyset) = 0$ and

$$\sum_{A \in \Im} m(A) = 1. \qquad (1)$$

$\Im$ is called the support of the random set, the sets A_i are the focal elements ($A_i \subseteq X$) and m is called the basic probability assignment. Each set, $A \in \Im$, contains some possible values of the variable, x, and $m(A)$ can be viewed as the probability that A is the range of x. Because of the imprecise nature of this formulation it is not possible to calculate the 'precise' probability Pro of a generic $x \in X$ or of a generic subset $E \subset X$, but only lower and upper bounds on this probability (Fig. 1): $Bel(E) \leq Pro(E) \leq Pl(E)$. In the limiting case, when ¡ is composed of single values only (singletons), then $Bel(E) = Pro(E) = Pl(E)$ and m is a probability distribution function.

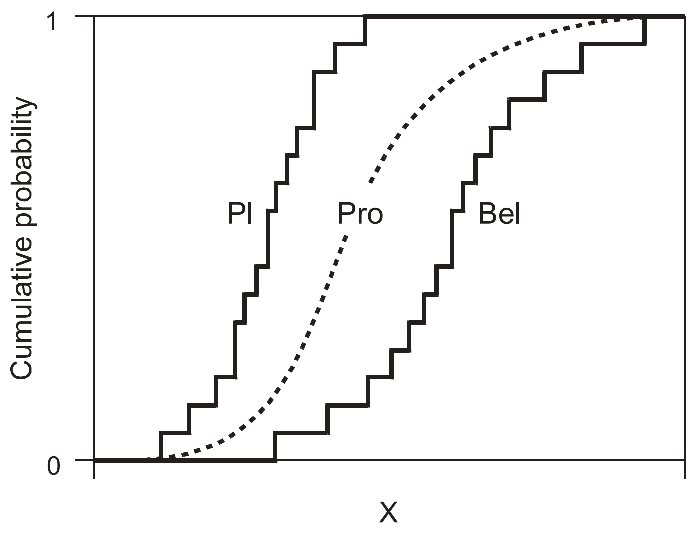

Figure 1: Upper bound *(Pl)* and lower bound *(Bel)* on 'precise' probability *(Pro)*

According to Dempster (1967) and Shafer (1976) the lower bound Bel (belief function) and the upper bound Pl (plausibility function) of its probability measure are defined, for every subset $E \in X$, by

$$Bel(E) = \sum_{A_i : A_i \subseteq E} m(A_i) \qquad (2)$$

$$Pl(E) = \sum_{A_i : A_i \cap E \neq \emptyset} m(A_i) \qquad (3)$$

which are envelopes of all possible distribution functions compatible with the data.

Bounds on the system response

Tonon *et al.* (2000a) showed that the extension of random sets through a functional relation is straightforward. Let f be a mapping $X_1 x \ldots x X_N \rightarrow Y$ and $x_1,\ldots,x_N$ be variables whose values are incompletely known. The incomplete knowledge about $x = (x_1,\ldots,x_N)$ can be expressed as a random relation R, which is a random set $(\Im, m)$ on the Cartesian product $X_1 x \ldots x X_N$. The random set $(\Re, \rho)$, which is the image of $(\Im, m)$ through f is given by (Tonon *et al.* 2000b):

$$\Re = \{R_j = f(A_i), A_i \in \Im\}; \qquad f(A_i) = \{f(\mathbf{x}), \mathbf{x} \in A_i\} \tag{2}$$

$$\rho(R_j) = \sum_{A_i : R_j = f(A_i)} m(A_i) \tag{3}$$

If $A_1,\ldots,A_n$ are sets on $X_1 x \ldots x X_N$ respectively and $x_1,\ldots,x_N$ are stochastically independent, then the joint basic probability assignment is the product measure given by

$$m(A_1 \times \ldots \times A_N) = \prod_{i=1}^{n} m_i(A_i), \quad A_1 \times \ldots \times A_n \in \Re \tag{6}$$

If the focal set A_i is a closed interval of real numbers: $A_i = \{x \mid x \in [l_i, u_i]\}$, then the lower and upper cumulative probability distribution functions, $F_*(x)$ and $F^*(x)$ respectively, at some point x (Fig. 2) can be obtained as follows:

$$F_*(x) = \sum_{i: x \geq u_i} m(A_i) \tag{7}$$

$$F^*(x) = \sum_{i: x \geq l_i} m(A_i). \tag{8}$$

In the absence of any further information, a random relation (random set model) can be constructed by assuming stochastic independence between marginal random sets (Equ. 6).

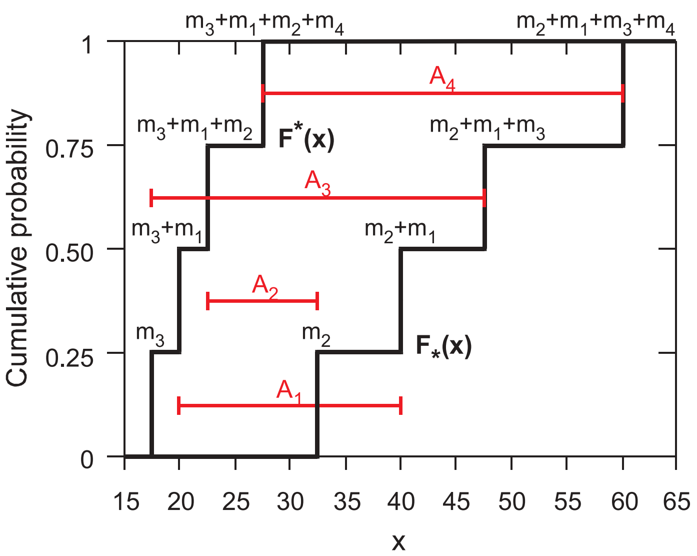

Figure 2: Focal sets Ai and upper and lower cumulative distribution function, $F_*(x)$ and $F^*(x)$.

The basic step is the calculation by means of Equation 4 and 5 of the image of a focal element through function f. The requirement for optimisation to locate the extreme elements of each set $R_j \in \Re$ (Equ. 4) can be avoided if it can be shown that the function $f(A_i)$ is continuous in all $A_i \in \Im$ and also no extreme points exist in this region, except at the vertices, in which case the Vertex method (Dong & Shah 1987) applies. Assume each focal element A_i is a N-dimensional box, whose 2^N vertices are indicated as vk, k = 1,…,2N. If the vertex method applies then the lower and upper bounds R_j* and R_j^* on each element $R_j \in \Re$ will be located at one of the vertices:

$$R_{j*} = \min_k \{f(v_k) : k = 1,\ldots 2^N\} \tag{9}$$

$$R_j^* = \max_k \{f(v_k) : k = 1,\ldots 2^N\} \tag{10}$$

Thus function $f(A_i)$ which represents in this framework a numerical model has to be evaluated 2^N times for each focal element A_i where N is the number of basic variables. The computational effort involved can be reduced significantly if $f(A_i)$ is continuous and a strictly monotonic function with respect to each parameter $x_1,\ldots,x_N$. In this case the vertices where the lower and upper bounds (Equ. 9 and 10) on the random set are located can be identified simply by consideration of the direction of increase of $f(A_i)$ which can be done by means of a sensitivity analysis (Schweiger & Peschl 2004). Thus $f(A_i)$ has to be calculated only twice for each focal element A_i (Tonon *et al.* 2000b).

Combination of random sets

An appropriate procedure is required if more than one source of information is available for one particular parameter in order to combine these sources. Suppose there are n alternative random sets describing some variable x, each one corresponding to an independent source of information. Then for each focal element $A \in X$

$$m(A) = \frac{1}{n} \sum_{i=1}^{n} m_i(A) \tag{11}$$

Alternative combination procedures have been proposed depending on different beliefs about the truth of the various information sources (e.g. Sentz & Ferson 2002, Hall & Lawry 2004).

The reliability problem

Basically, reliability analysis calculates the probability of failure $p_f = p(g(x) \leq 0)$ of a system characterised by a vector $x = (x_1,\ldots,x_N)$ of basic variables on X where g is called the 'limit state function' and pf is identical to the probability of limit state violation. Utilising random set theory the reliability problem is reduced to evaluate the bounds on p_f subject to the available knowledge restricting the allowed values of x. If the set of failed states is labelled $F \subseteq X$, the upper and lower bound on the probability of failure are the Plausibility $Pl(F)$ and Belief $Bel(F)$ respectively: $Bel(F) \leq p_f \leq Pl(F)$.

THE RANDOM SET FINITE ELEMENT METHOD

The assessment of the stability of a geotechnical system is based on various sources of information such as ground conditions, construction procedure and others. Although it is well established that there is a significant scatter e.g. in the material parameters a deterministic approach with design values 'on the safe side' or a parametric study based on experience is commonly adopted. Sometimes worst case assumptions are postulated. However they are not always obvious, in particular for complex, highly nonlinear systems such as geotechnical structures. By using finite element codes in reliability analysis there are some advantages compared with limit equilibrium methods or other similar methods because more than one system parameter can be obtained without the need of changing the computational model. These can be used for the evaluation of the serviceability or the ultimate limit state of a geotechnical system or of the respective elements of the system. Examples are displacements in the soil or stresses in structural elements.

The proposed method combines random sets to represent uncertainty and the finite element analysis (RS-FEM: random set finite element method). Figure 3 depicts an overview for a typical random set finite element calculation in geotechnics (basic procedure).

APPLICATION TO DEEP EXCAVATION PROBLEM

In order to demonstrate the applicability of the proposed method some results from a back analysis of a practical example, namely a deep excavation (Breymann et al. 2003) on a thick layer of post-glacial soft lacustrine deposit (clayey silt) are presented. An underground car park has been constructed as open pit excavation with a 24m deep anchored concrete diaphragm wall as retaining construction. In addition a berm was left before excavating to full depth and the foundation slab in the centre of the excavation was constructed in sections. Figure 4 plots the cross section of the system and the soil profile. In this analysis particular attention is given to the assessment of the angular distortion of the building exceeding acceptable limits.

Subsoil conditions and material parameters

The behaviour of the subsoil is characterised by soil parameters established from a number of laboratory and in situ tests. In order to assess the applicability of the proposed approach in practical design situations only data available before the excavation started has been used. Of particular significance for the deformation behaviour of the soft-plastic clayey silt is the deformation modulus Es (denoted as Eoed in the following), gained from one-dimensional compression tests on undisturbed soil samples after pre-loading with the in situ stress of the relevant depth. The parameters used in the analysis performed with the finite element code PLAXIS V8 (Brinkgreve 2000) are summorised in Tables 1 to 3. The mesh consists of approximately 1.400 15-noded triangular elements and the Hardening-Soil model (HS) was used to model the behaviour of the different soil layers.

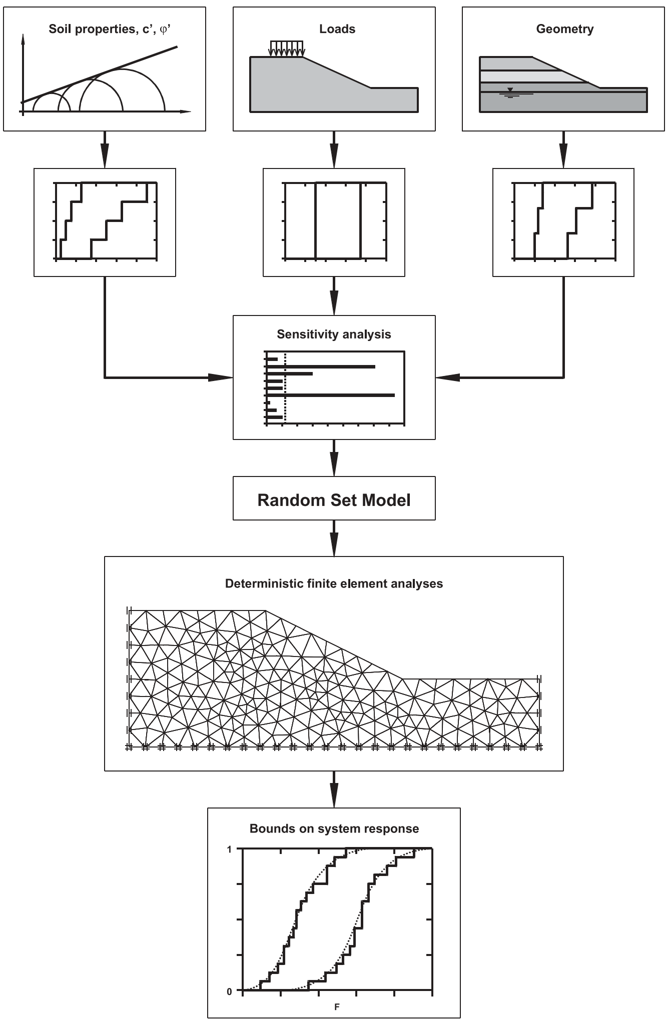

Figure 3: Schematic representation of a typical random set finite element calculation

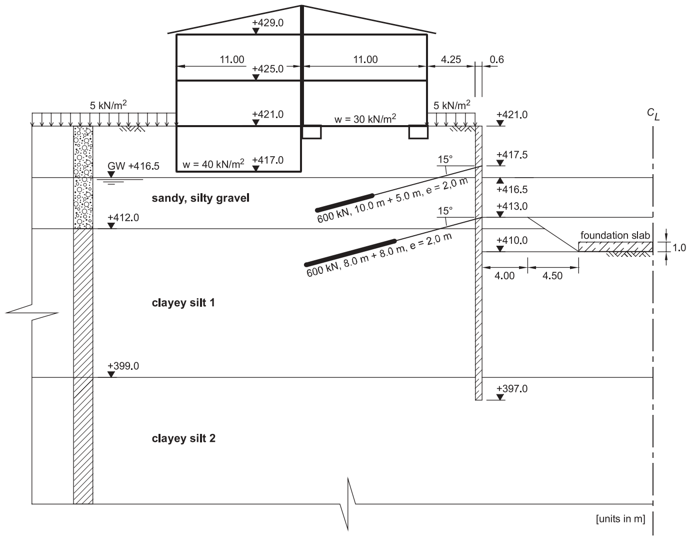

Figure 4: Geometry and subsoil conditions

Soil	level	type	γ	γ_{Sat}	k_x	k_y
	m		kN/m³	kN/m³	m/d	m/d
Sandy, silty gravel	421.0-412.0	drained	18.9	20.7	2.59	2.59
Clayey silt 1	412.0-399.0	undrained	20.0	20.2	8.6E-3	8.64E-5
Clayey silt 2	399.0-370.0	undrained	20.0	20.2	8.6E-3	8.64E-5

Table 1: Depth of layers and permeabilities

Soil	E_{oed}^{ref*}	E_{50}^{ref*}	E_{ur}^{ref*}	ν_{ur}	m	k_o^{nc}	p_{ref}
	kN/m²	kN/m²	kN/m²	–	–	–	kN/m²
Sandy, silty gravel	30 000	30 000	90 000	0.20	0	0.426	100
Clayey silt 1	25 000	25 000	75 000	0.20	0.3	0.562	100
Clayey silt 2	35 000	70 000	210 000	0.20	0.8	0.562	100

Note. * = Basic variables for random set model

Table 2: Stiffness parameters (reference values)

Soil	$c*$	$\varphi*$	ψ
	kN/m²	°	°
Sandy, silty gravel	2.0	35.0	5.0
Clayey silt 1	25.0	26.0	0
Clayey silt 2	25.0	26.0	0

Note. * = Basic variables for random set model

Table 3: Strength parameters (reference values)

It should be noted that stiffness and strength parameters in Table 2 and 3 are the ones applied as reference values. Structural elements such as walls and foundation slabs have been modelled as linear elastic material, whereas the concrete diaphragm wall and the anchors have been modelled as elasto-plastic material in order to arrive at a more realistic failure mechanism for the ultimate limit state (Tab. 4).

Structural element	type	EA	EI	c	φ	T_{pl}
		kN/m	kNm²/m	kN/m²	°	kN/m²
Diaphragm wall	elasto-plastic	180	5.4	3 963	40.0	1.9E4
Foundation slab	elastic	2.2E7	1.173E6	–	–	–
Walls	elastic	2.2E7	1.173E6	–	–	–
Anchors	elasto-plastic	8.4E4	–	–	–	3.3E5

Table 4: Parameters for structural elements

Basic variables for the random set model
The material parameters for the soil layers, which were treated as basic variables are summerised in Table 5. The parameters have not been determined solely from experimential investigations (geotechnical report) but also from previous experience of finite element analyses and in situ measurements under similar conditions (expert knowledge). The table highlights the wide range of certain parameters which in itself to some degree contain engineering judgement of people involved, e.g. a significant capillary cohesion has been assigned to the sandy, silty gravel in the geotechnical report.

Soil	*c*	φ'	E_s
Information source	kN/m²	°	MN/m²
Sandy, silty gravel			
Geotechnical report	0 - 50.0	35.0 - 37.0	20.0 - 35.0
Expert knowledge	0 - 5.0	34.0 - 36.0	30.0 - 50.0
Clayey silt 1			
Geotechnical report	0 - 20.0	22.0 - 30.0	5.0 - 25.0
Expert knowledge	10.0 - 30.0	24.0 - 28.0	20.0 - 40.0
Clayey silt 2			
Geotechnical report	0 - 20.0	22.0 - 29.0	20.0 - 30.0
Expert knowledge	10.0 - 30.0	24.0 - 28.0	30.0 - 40.0

Table 5: Basic variables for material parameters (input values)

Soil properties do not vary randomly in space but their variation is gradual and follows a pattern that can be quantified using spatial correlation structures whereas the use of perfectly correlated soil properties gives rise to unrealistically large values of failure probabilities for geotechnical structures (see e.g. Mostyn & Soo 1992 and Fenton & Griffiths 2003). PLAXIS requires the soil profile to be modelled using homogeneous layers with constant soil properties. Due to the fact of spatial averaging of soil properties the inherent spatial variability of a parameter, as measured by its standard deviation, can be reduced significantly. The variance of these spatial averages can be correlated to the point variance using the variance reduction technique. In this framework the variance reduction technique by Vanmarke (1983) is applied which depends on the averaging volume described by the characteristic length and the type of correlation structure. The approach followed here (Peschl & Schweiger 2004) is certainly a simplification compared to real field behaviour. However, a more rigorous treatment of the spatial correlation requires computational efforts which are not feasible in practical geotechnical engineering at the present time, at least not in combination with high level numerical modelling.

For the material parameters given in Table 5 two published sources of information were available and these interval estimates were combined using the averaging procedure in Equation 11. As an example the random set for the effective friction angle φ' of the gravel layer is depicted in Figure 5. Most values for the vertical spatial correlation length for sandy gravel materials and clayey silt deposits recorded in the literature are in the range of about 0.5 up to 5.0m. Therefore, a value of about 2.5m is assumed in this case. The characteristic length has been taken as 55m, which is based on analyses investigating potential failure mechanisms for this problem.

Construction steps modelled

The analyses performed were calculated as 2D plane strain problems and do not consider 3D-effects. It could be reasonably assumed that consolidation effects do not play a significant role for the excavation-induced movements and therefore an undrained analysis in terms of effective stresses was performed for the clayey silt layers. The computational steps have been defined according to the real construction process.

CALCULATION RESULTS

Before the random set analysis as described previously is performed a sensitivity analysis quantifying the influence of each variable on certain results can be made (Schweiger & Peschl 2004). For the 9 variables shown in Table 5, 37 calculations are required to obtain a sensitivity score for each variable. In this case the horizontal displacement of the top of the diaphragm wall, ux, the angular distortion, d/l, of the adjacent building at each construction step and the safety factor, determined by means of the j/c-reduction technique, at the final construction step is evaluated. Based on the results of the sensitivity analysis the following parameters were considered in the random set model: cohesion for the sandy, silty gravel layer and the stiffness parameters E_{oed}, E_{50} and E_{ur} (but these are correlated) for the sandy, silty gravel and the upper clayey silt layer, i.e. 64 calculations are required.

Serviceability limit state

The angular distortion d/l of a structure, with d being the differential settlement and l the corresponding length, is often used as measure to assess the likelihood of damage. The value tolerated depends on a number of factors such as the structure of the building, the mode of deformation and of course the purpose the building has to serve. A ratio of about 1:600 is used here as a limiting value for the evaluation of the limit state function in order to obtain the reliability in terms of serviceability.

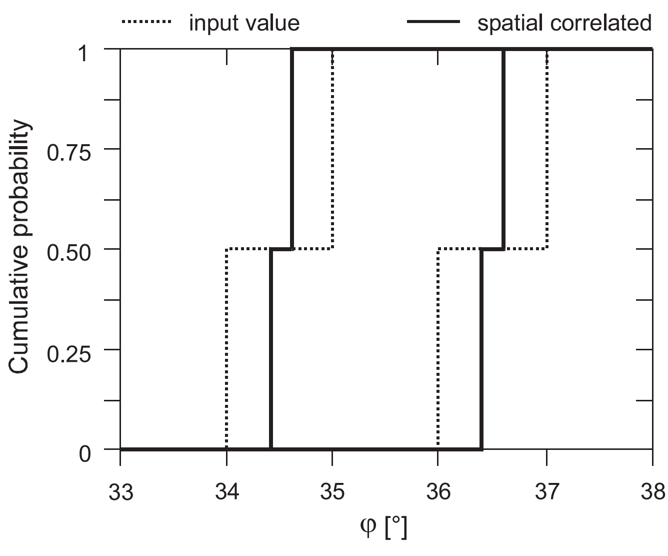

Figure 5: Random set for the friction angle of the gravel layer

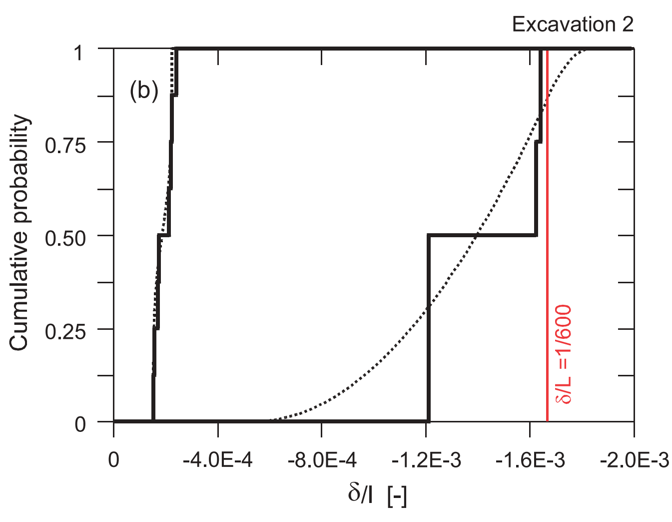

Figure 6: Range of angular distortion d/l after second excavation step

As an example Figure 6 depicts the calculated bounds (CDF's) of the angular distortion d/l after the second excavation step. These discrete CDF's were fitted using best-fit methods like the method of least squares in order to achieve a continuous function (dotted line in Fig. 6), which are used for the evaluation of the limit state function by means of Monte-Carlo simulations. By doing so, results in ranges on the probability of failure are obtained as given in Table 6 for all construction steps. The probabilities show that damages of the adjacent building can be expected already during the second excavation step (crucial construction step) and continues to be critical throughout the following construction steps.

Construction step	Fitted distribution		max p_f	min p_f
	Upper bound	Lower bound	-	-
Excavation 1	Beta	Gamma	0	0
Anchor 1	Gamma	Beta	3.0E-5	0
Excavation 2	Triangular	Beta	1.3E-1	0
Anchor 2	Gamma	Beta	2.2E-1	0
Excavation 3	Exponential	Beta	7.2E-1	0
Excavation 4	Beta	Beta	9.7E-1	0

Table 6. Range of probability that $\delta/l \geq 1/600$

ULTIMATE LIMIT STATE

For the evaluation of the ultimate limit state the shear strength reduction technique is applied. In this study, the diaphragm wall and the anchors have been modelled as an elasto-plastic material (Tab. 4) as suggested by Schweiger (2003). This means that for the evaluation of the factor of safety via φ/c-reduction not only the shear strength parameters of the soil layers but also the strength parameters of the wall are successively reduced until a failure mechanism is developed, taking into account an increase in anchor forces until their ultimate capacity is reached. The range of the resulting factor of safety and a deformed mesh of the final excavation step after φ/c-reduction is shown in Figure 7 and Figure 8.

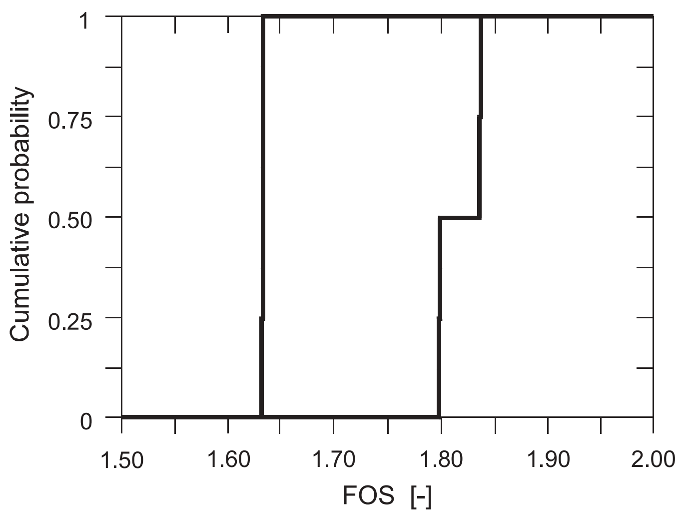

Figure 7: Range of factor of safety after φ/c-reduction of final excavation step

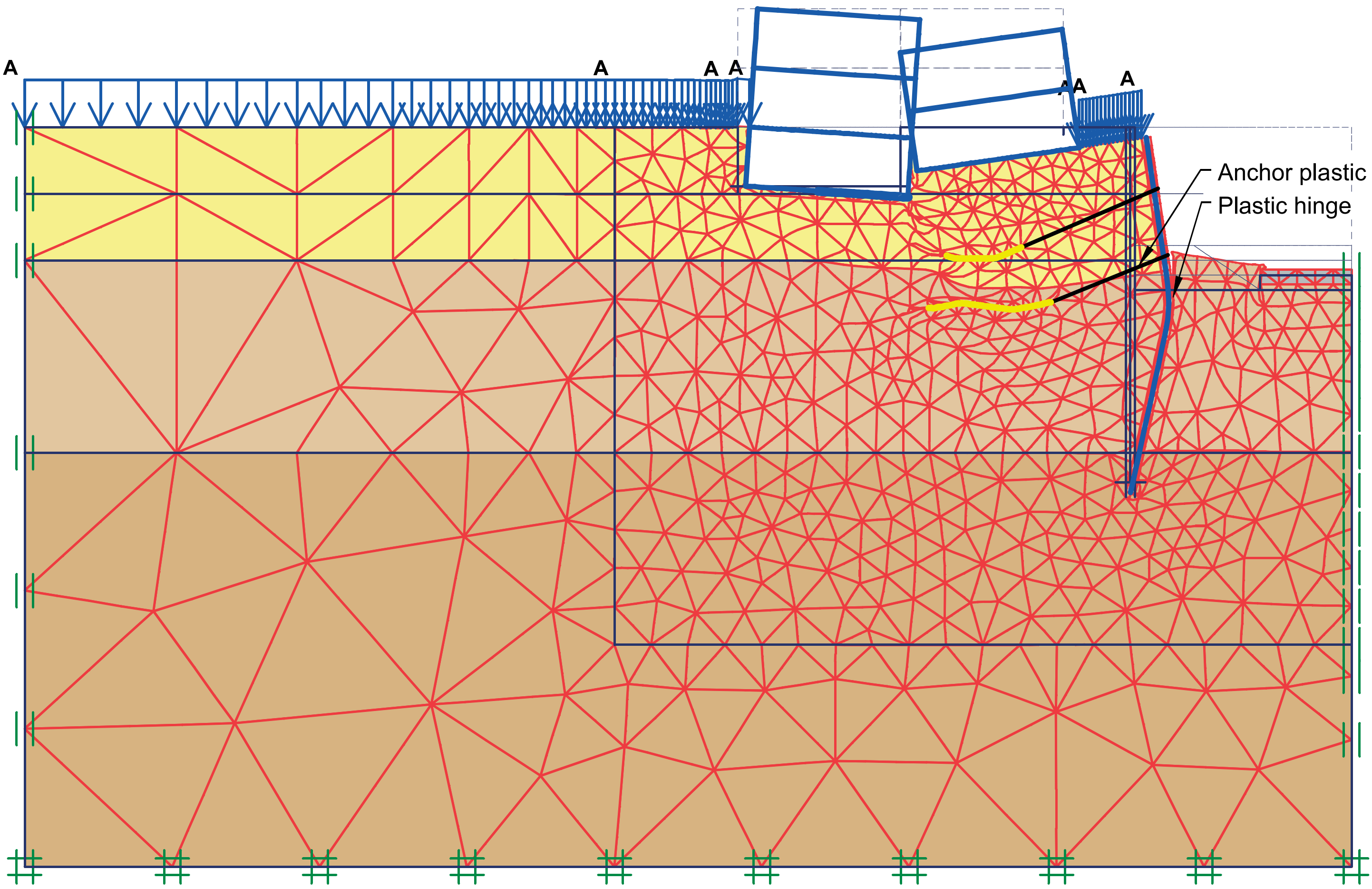

Figure 8: Deformed mesh obtained by φ/c-reduction after final excavation step

CONCLUSIONS

Reliability analysis in engineering conventionally represents the uncertainty of the system state variables as precise probability distributions and applies probability theory to generate precise estimates of e.g. the probability of failure or the reliability. However, it has been recognised that traditional probability theory cannot capture the full scope of uncertainty (inherent variability and lack of knowledge). The presented approach offers an alternative way of analysis when insufficient information is available for treating the problem by classical probabilistic methods. It requires less computational effort but has the disadvantage that spacial correlation can be taken into account only in a simplified way by means of variance reduction factors. The applicability of the proposed method for solving practical boundary value problems has been shown by analysing the excavation sequence for a deep excavation in soft soil.

The significant innovation of the presented framework is that it allows for the allocation of a probability mass to sets or intervals and provides a consistent framework for dealing with uncertainties throughout the design and construction of a project, because the model can be refined by adding more information when available without changing the underlying concept of analysis. As a side effect worst case assumptions in terms of unfavourable parameter combinations have not to be estimated from experience but are automatically generated. The argument that engineering judgement will do the same much faster is not entirely true because in complex non-linear analyses, which become more and more common in practical geotechnical engineering, the parameter set for a highly advanced constitutive model leading to the most unfavourable result in terms of serviceability and ultimate limit state for all construction stages is not easy to define. From a practical point of view the concept of dealing with 'ranges' seems to be more appealing for engi-neers than working in density distribution functions.

REFERENCES

- Ang, A. H-S. & Tang, W.H. (1975). Probability Concepts in Engineering Planning and Design. Volume I-Basic Principles. New York: John Wiley & Sons.
- Breymann, H., Moser, M. & Premstaller, M. (2003). Die Gründung des Kongresshauses Salzburg – Konsequenzen aus dem Verformungsverhalten des Salzburger Seetons. Felsbau 21, No. 5, Essen: VGE, 10-17.
- Brinkgreve, R. B. J. (2000). PLAXIS, Finite element code for soil and rock analyses. Users manual, Rotterdam: Balkema.
- Dempster, A. P. (1967). Upper and lower probabilities induced by a multivalued mapping. Annals of Mathematical Statistics 38, 325-339.
- Dong, W. & Shah, H. C. (1987). Vertex method for computing functions of fuzzy variables. Fuzzy Sets & Systems 24, 65-78.
- Dubois, D. & Prade, H. (1990). Consonant approximation of belief functions. Int. J. of Approximate Reasoning 4, 419-449.
- Dubois, D. & Prade, H. (1991). Random sets and fuzzy interval analysis. Fuzzy Sets and Systems 42, 87-101.
- Fenton, G. A. & Griffiths, D. V. (2003). Bearing-capacity prediction of spatially random c-j soils. Canadian Geotechnical Journal, Vol. 40, No. 1, 54-65.
- Hall, J. W. & Lawry, J. (2004). Generation, combination and extension of random set approximations to coherent lower and upper probabilities. Reliability Engineering & System Safety 85, No. 1-3, 89-101.
- Helton, J. C. (1997). Uncertainty and Sensitivity Analysis in the Presence of Stochastic and Subjective Uncertainty. Journal of Statistical Computation and Simulation 57, 3-76.
- Mostyn, G. R. & Soo, S. (1992). The effect of autocorrelation on the probability of failure of slopes. Proc. Sixth ANZ Conference on Geomechanics, 542-546.
- Peschl, G. M. & Schweiger, H. F. (2004). Application of the random set finite element method (RS-FEM) in geotechnics. Proc. of the 9th Symposium on Numerical Models in Geomechanics - NUMOG IX, Ottawa, 249-255.
- Schweiger, H. F. (2003). Standsicherheitsnachweise fuer Boeschungen und Baugruben mittels FE-Methode durch Abmin-derung der Scherfestigkeit. Workshop AK 1.6, Schriftenreihe Geotechnik, 11, Bauhaus-Universitaet Weimar, 19-36.
- Schweiger, H. F. & Peschl, G. M. (2004). Numerical analysis of deep excavations utilizing random set theory. Proc. Geotechnical Innovations, Essen: VGE, 277-294.
- Sentz, K. & Ferson, S. (2002). Combination of evidence in Dempster-Shafer theory. Report SAND2002-0835, Sandia Nat. Laboratories, Albuquerque, New Mexico.
- Shafer, G. (1976). A Mathematical Theory of Evidence. Princeton: Princeton University Press.
- Tonon, F., Bernardini, A. & Mammino, A. (2000a). Determination of parameters range in rock engineering by means of Random Ret Theory. Reliability Engineering and System Safety 70, No. 3, 241-261.
- Tonon, F., Bernardini, A. & Mammino, A. (2000b). Reliability analysis of rock mass response by means of Random Set Theory. Reliability Engineering and System Safety 70, No. 3, 263-282.
- Vanmarcke, E. H. (1983). Random Fields – Analysis and Syntesis. Cambridge, Massachusetts: MIT-Press.
- Wright, G. & Ayton P. (1994). Subjective Probability. New York: Wiley.