SVSLOPE (SEISMIC)

TECHNICAL PREVIEW

2D FINITE ELEMENT
EARTHQUAKE ANALYSIS

Theory Manual

Written by:
The Bentley Systems Team

Last Updated: Tuesday, August 04, 2020

Bentley Systems Incorporated

1 INTRODUCTION

SVSLOPE (SEISMIC) is currently in Technical Preview.

Seismic analysis of geotechnical structures involves finding the solution of a set of time-dependent Partial Differential Equations (PDEs) that describe the response of the geotechnical system to a high frequency dynamic excitation, i.e., an earthquake. Earthquake-induced stresses that can be used for stability analysis are also an outcome of seismic analysis. Simplified pseudo-static analysis incorporates the effect of inertial forces into static analysis to provide a valuable insight into some peak values, such as deformations and stresses, that are essential for design. However, many important aspects of a seismic analysis will remain unrevealed unless a fully dynamic analysis, that solves the time-dependent PDEs, is performed. The propagation and interaction of seismic waves and the variation of stresses over time can only be captured by means of a dynamic analysis.

SVSLOPE (SEISMIC) is a finite element analysis package capable of solving two-dimensional dynamic stress-strain models. SVSLOPE (SEISMIC) performs time-domain dynamic analysis under various types of user-defined dynamic loads. The package is capable of performing seismic analysis using imported earthquake records as well as a combination of dynamic forces and seismic excitations. The availability of a vast choice of boundary conditions provides flexibility for simulating different field conditions such as free surfaces, continuous domains, rigid and semi rigid boundaries, etc. Our user-friendly and efficient interface allows the user to conveniently input the geometry, assign material properties, and define dynamic loads and seismic boundary conditions.

SVSLOPE (SEISMIC) is a module within the SOILVISION software; therefore, geometry can easily be transferred from other modules to SVSLOPE (SEISMIC). The output generated by SVSLOPE (SEISMIC) can be imported to the SVSLOPE module to perform dynamic slope stability analysis for two-dimensional models. SVSLOPE (SEISMIC) uses the SOILVISION front-end interface for model setup, SOILVISION dynamic finite-element engine for the numerical analysis, and the SOILVISION back-end interface for results visualization.

This manual provides the theoretical basis and the computational approach used by SVSLOPE (SEISMIC) for performing seismic analysis. The manual, however, does not intend to present an exhaustive review of all theories and numerical techniques associated with seismic analysis. Appropriate citations to the relevant references are provided throughout this manual and the user may refer to those references, when in-depth discussions of the theory and numerical techniques are required. For details regarding the software operation and modeling guidelines the user should consult the USER MANUAL and the SVSLOPE (SEISMIC) TUTORIAL MANUAL, included with the software.

This documentation is organized in the following chapters:

- Chapter 1 presents an overview of SVSLOPE (SEISMIC)
- Chapter 2 explains the basic principles and concepts involved in the formulation of the governing equations for a general dynamic analysis in SVSLOPE (SEISMIC)
- Chapter 3 presents the stress-strain relationships and provides information on the selection of appropriate material properties
- Chapter 4 discusses the numerical schemes implemented in SVSLOPE (SEISMIC) for solving time-dependent systems of PDEs
- Chapter 5 explains the initial conditions and their implementation in SVSLOPE (SEISMIC)

- Chapter 6 explains the theory and implementation of dynamic boundary conditions in SVSLOPE (SEISMIC)
- Chapter 7 presents a useful list of references and reading materials

1.1 ANALYSIS TYPES

SVSLOPE (SEISMIC) can handle the following types of analyses:

Stress variable: Total stress

Stress/strain conditions: (a) 2D plane strain
(b) General 3D stress conditions.

1.2 CONSTITUTIVE MODELS

SVSLOPE (SEISMIC) currently allows total stress analysis using a linear elastic constitutive law. More advanced material constitutive models will be added to the SVSLOPE (SEISMIC) library of constitutive models in the near future.

1.3 BOUNDARY CONDITIONS

Several types of essential and natural boundary conditions can be defined in SVSLOPE (SEISMIC). The vast range of available boundary conditions allow for the enforcement of general fixed and free conditions as well as prescribed values of displacement, velocity, or acceleration at any boundary of the domain. Different types of non-reflecting boundary conditions can also be defined to simulate truncated domains in a continuum. SVSLOPE (SEISMIC) is also capable of performing dynamic analysis under various combinations of dynamic loads and seismic motions.

2 BASIC PRINCIPLES AND EQUATIONS

SVSLOPE (SEISMIC) is based on the dynamic analysis theory, applied to two-dimensional (i.e., plane-strain analysis) and three-dimensional models. Dynamic stress analysis can be expressed in terms of time dependent partial differential equations, governing the motion of an infinitesimal differential volume of material.
The equations of motion are combined with constitutive relations and strain-displacement equations to produce a displacement-based formulation. SVSLOPE (SEISMIC) employs the infinitesimal strains theory for this purpose. Choices of spatial and temporal numerical integration schemes are also required for solving the final set of differential equations, which will be explained in Chapter 4 of this manual.

This chapter explains the basic principles and conventions adopted in SVSLOPE (SEISMIC) for formulating the governing equations of a general (seismic) dynamic analysis.

2.1 STRESS/STRAIN CONDITIONS AND COORDINATE SYSTEMS

Dynamic behavior of a mechanical system is governed by a set of time-dependent PDEs solution of which requires the evaluation of spatial and temporal derivatives. The choice of a spatial coordinate system and a time dimension directly affects the solution of the governing equations. Hence, this section explains the conventions adopted in SVSLOPE (SEISMIC) for the spatial coordinate system and time dimension.

2.1.1 Spatial Coordinate System

A two-dimensional Cartesian coordinate system (*x, y*) is used to present the geometry of a 2D plane strain model where the *y*-direction corresponds to the elevation:

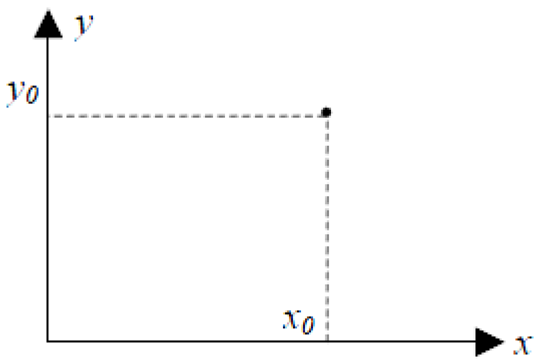

Figure 1 2D Cartesian coordinate system for 2D plane strain analysis

Two-dimensional plane strain problems correspond to conditions where the strain in the *z*-direction is zero. Therefore, the stress and strain distributions are independent of the *z*-coordinate. In other words, there is a cross-section of the problem geometry in the *x*-*y* plane that represents the entire problem. Many geotechnical problems can be approximated by a plane strain analysis. Examples of such problems include earth embankments, strip footings, highway embankments, and pipeline foundations.

For general 3D models, a 3D cartesian coordinate system (*x, y, z*) is adopted for presenting the geometry of the model. The z axis corresponds to the elevation in this case.

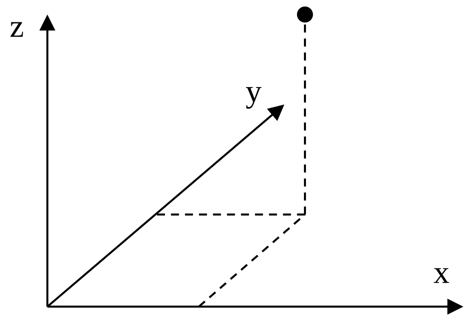

Figure 2 3D Cartesian coordinate system for general 3D analysis

The geometry and other conditions of a particular application usually suggests the necessity of using 3D analysis or the possibility of using a simplified 2D model. It is worth mentioning that problems that do not completely conform with the plane strain condition are still often analyzed using the plane strain approach, by selecting representative cross sections. Such approximation is considered common in geotechnical engineering practice but requires engineering judgement.

2.1.2 Time Dimension

The time domain in dynamic analysis is a one-dimensional domain with positive values. The time origin (i.e., $t = 0$) in SVSLOPE (SEISMIC) is assumed to be the time at which a dynamic analysis starts. The initial conditions of the dynamic analysis should be defined at this time.

2.2 SIGN CONVENTIONS

SVSLOPE (SEISMIC) uses the sign conventions that are most commonly used in geotechnical engineering. The sign conventions adopted are specified in the following sections.

2.2.1 Load and Displacement

The sign convention of loads and displacements are defined according to the positive directions of the global coordinate system introduced in the previous section. A force component is considered positive if it is acting in one of the positive directions of the global coordinate system. Similarly, a displacement component that occurs in a positive direction of the global coordinate system is considered positive. For example, a positive force acting in the *x*-direction will be acting towards the right. An upward displacement in a 2D problem will be considered a positive displacement in the *y*-direction. A downward force or displacement in a 3D model will be considered a negative force or displacement in the *z*-direction.

2.2.2 Stresses

SVSLOPE (SEISMIC) follows the standard geotechnical engineering sign convention where compression is positive. To maintain the generality of the topic, in this manual, stress analysis is discussed in the general 3D form and reduced to 2D plane strain conditions when necessary. Consider an infinitesimal differential volume of material, as illustrated in Figure 3. A surface of the differential volume is considered a positive surface if its outward normal vector points in the positive coordinate direction. Similarly, negative surfaces are those whose normal vectors point in the negative coordinate direction. Figure 3(a) illustrates all positive and negative surfaces of a differential volume.

The conventions of positive stress components are presented in Figure 3(b). On a positive surface, all stresses that act in the negative coordinate directions are considered positive. On a negative surface, all stresses acting in the positive coordinate directions are positive. Therefore, compression is positive, and tension is negative. The above rule also applies to shear stresses.

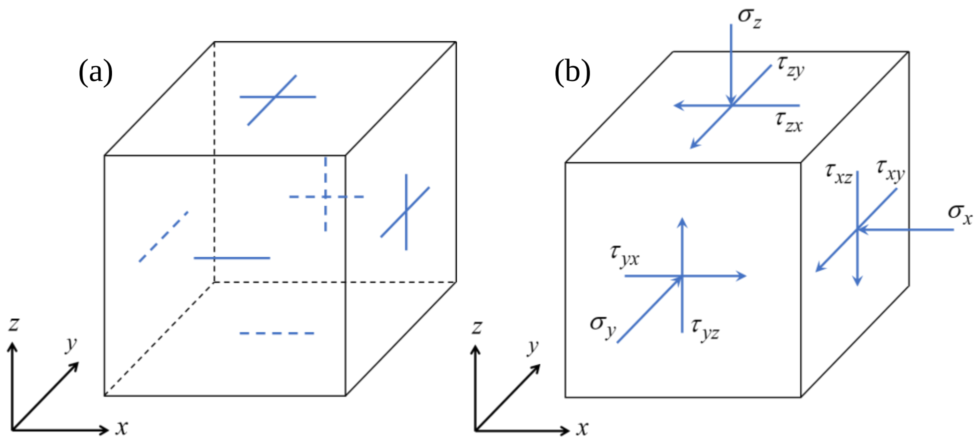

Figure 3 Illustration of the convention for (a) positive and negative surfaces and (b) positive components of stress tensor in SVSLOPE (SEISMIC)

2.2.3 Strains

Normal strains are considered positive when the length in the direction of the strain decreases. When a differential volume is subject to positive shear strains, the right angles between its edges and the extensions of the coordinate axes in the negative directions increase, while the right angles between the positive directions of the coordinate axes and the extensions of the edges in the negative directions decrease.

2.3 UNITS

Seismic analysis in SVSLOPE (SEISMIC) can be performed in either Metric or Imperial units. The system of units selected dictates the required units when entering material properties, loads, and geometric dimensions.

The following table presents model variables and their units in both Metric and Imperial systems:

Table 1 Model Variable Units

VARIABLE	METRIC SYSTEM	IMPERIAL SYSTEM
Distance	meter, *m*	foot, *ft*
Time	second, *s*	second, *s*
Displacement	meter, *m*	foot, *ft*
Velocity	*m/s*	*ft/s*
Acceleration	m/s^2	ft/s^2

Strain	%	%
Force	kilo Newton, *kN*	pound force, *lbf*
Concentrated load	kilo Newton, *kN*	pound force, *lbf*
Spread/line load	*kN/m*	*lbf/ft*
Stress	kN/m^2	lbf/ft^2 (*psf*)
Acceleration of gravity	m/s^2	ft/s^2
Mass	kilogram, *kg*	pound mass (pound), *lbm* (*lb*)
Unit weight	kN/m^3	lbf/ft^3 (*pcf*)
Young modulus	*kPa*	lbf/ft^2 (*psf*)
Poisson ratio	unitless	unitless
Damping ratio	unitless	unitless
Frequency	hertz, *Hz* (s^{-1})	hertz, *Hz* (s^{-1})
Phase	radian, *rad*	radian, *rad*

2.4 STRESS AND STRAIN TENSORS

Considering Cartesian coordinates, the general stress and strain tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ respectively, for a total stress analysis are presented as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{bmatrix} \qquad [1]$$

where:

σ_i and ε_j = respectively the normal stress and strain acting on the *i*-plane, in the *i*-direction ($i, j \in \{x, y, z\}$)

τ_{ij} and γ_{ij} = respectively the shear stress and strain acting on the *i*-plane, in the *j*-direction ($i, j \in \{x, y, z\}$)

For 2D plane strain conditions, the normal strains, shear strains, and shear stresses acting in the *z*-direction vanish, and the stress and strain tensors reduce to the following:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & 0 \\ \gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [2]$$

2.5 DISPLACEMENT DERIVATIVES

2.5.1 Spatial Derivatives

In general 3D conditions, the relationships between the components of strain and displacements can be written, in a Cartesian coordinate system, as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x}\ ,\quad \varepsilon_y = \frac{\partial v}{\partial y}\ ,\quad \varepsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},\quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad [3]$$

where:

u, v, z = the displacements in the x-, y-, and z-directions, respectively

For 2D plane strain conditions, the z component of displacement and all derivatives with respect to z-coordinates are zero. Hence, the six components of strain reduce to the following three:

$$\varepsilon_x = \frac{\partial u}{\partial x}\ ,\quad \varepsilon_y = \frac{\partial v}{\partial y}\ ,\quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad [4]$$

These relationships are derived assuming small strains.

2.5.2 Temporal Derivatives

The first and second temporal derivatives of displacement in each direction, respectively represent the values of particle velocity and acceleration in that direction:

$$\dot{u} = \frac{\partial u}{\partial t}\ ,\quad \dot{v} = \frac{\partial v}{\partial t},\quad \dot{w} = \frac{\partial w}{\partial t}$$
$$\ddot{u} = \frac{\partial^2 u}{\partial t^2},\quad \ddot{v} = \frac{\partial^2 v}{\partial t^2},\quad \ddot{w} = \frac{\partial^2 w}{\partial t^2} \qquad [5]$$

where:

$\dot{u}$, $\dot{v}$, and $\dot{w}$ = the values of particle velocity in the x-, y-, and z-directions, respectively

$\ddot{u}$, $\ddot{v}$, and $\ddot{w}$ = the values of acceleration in the x-, y-, and z-directions, respectively

2.6 EQUATIONS OF MOTION

The motion of an infinitesimal differential element of a continuum under the applied stresses (stresses are illustrated in Figure 3(b)) is governed by the conservation of linear momentum (also known as the equation of motion). The differential element should be sufficiently small such that material properties and measured variables can be considered continuous within the element.

2.6.1 General 3D

The equations of motion in the *x*-, *y*-, and *z*-directions, respectively, are presented as follows:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+b_x=\rho_s\frac{\partial^2 u}{\partial t^2} \quad [6]$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+b_y=\rho_s\frac{\partial^2 v}{\partial t^2} \quad [7]$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+b_z=\rho_s\frac{\partial^2 w}{\partial t^2} \quad [8]$$

where:

σ_i and τ_{ij} = the total normal and shear stresses ($i, j\in\{x, y, z\}$), respectively,

b_i = the component of body force in the *i*-direction ($i\in\{x, y, z\}$)

The conservation of angular momentum suggests $\tau_{ij}=\tau_{ji}$. The terms on the right-hand side of [6]-[8] represent the inertial forces applied to the differential element, in which:

ρ_s = the average mass density of the solid in the differential element

t = analysis time

2.6.2 2D Plane Strain

By default, SVSLOPE (SEISMIC) creates all two-dimensional models in the *x-y* plane of the Cartesian coordinate system. Therefore, in a two-dimensional plane strain analysis, the component of displacement in the z-direction and all special derivatives with respect to the z-coordinate vanish. Thus, the equations of motion reduce to the following two equations in the *x*- and *y*-directions, respectively:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+b_x=\rho_s\frac{\partial^2 u}{\partial t^2} \quad [9]$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+b_y=\rho_s\frac{\partial^2 v}{\partial t^2} \quad [10]$$

in which, all variables have been defined previously.

The equations of motion, [9] and [10], are time dependent and represent the propagation of two coupled disturbances (waves) in the *x-y* plane. The longitudinal disturbance is known as the Dilatational or Primary wave (P-wave) and the transverse disturbance is known as the Shear or Secondary wave (S-wave). In the general 3D case, the equations of motion, [6]-[8], represent the propagation of one P-wave and two S-waves (known as SH- and SV-wave) in the three-dimensional space. For a detailed discussion about the wave theory, the user is referred to Harris (2001) and Achenbach (1973).

2.7 FINAL PARTIAL DIFFERNTIAL EQUATIONS

SVSLOPE (SEISMIC) uses a displacement-based dynamic formulation; therefore, the stress-based equations of motion should be converted to displacement-based Partial Differential Equations by means of the stress-strain constitutive relations.

Let the stress-strain relation in its general form be defined by:

$$\boldsymbol{\sigma} = \mathbf{S}(\boldsymbol{\varepsilon}) \qquad [\ 11\ ]$$

where:

$\mathbf{S}$ = the function that defines the stress-strain relation

This general representation results in an incremental form of the constitutive relation that can be used to express various types of constitutive models. The incremental form of the constitutive equation is obtained by differentiating [11] with respect to the strain tensor:

$$d\boldsymbol{\varepsilon} = \mathbf{D}^{-1} d\boldsymbol{\sigma} \qquad [\ 12\ ]$$

where:

$$\mathbf{D} = \frac{d\mathbf{S}(\boldsymbol{\varepsilon})}{d\boldsymbol{\varepsilon}} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{45} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}$$

$$\boldsymbol{\sigma}^T = \left\{ \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \right\}$$

$$\boldsymbol{\varepsilon}^T = \left\{ \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \right\}$$

The above generalized relationship is used in this section to present a general format for constitutive models implemented in SVSLOPE (SEISMIC). The user is referred to Chapter 3 for a detailed discussion of constitutive models that can be adopted by SVSLOPE (SEISMIC). The last step for converting the stress-based equations of motion to displacement-based PDEs is re-writing the components of strain in terms of displacements using the strain-displacement relations presented in section 2.5.1.

2.7.1 General 3D

Re-writing the components of strain in terms of displacements and substituting stress-strain relations into the equations of motion, the set of PDEs governing the motion of an infinitesimal differential element under general 3D conditions can be re-written as:

$$\frac{\partial}{\partial x}\left[D_{11}\frac{\partial u}{\partial x}+D_{12}\frac{\partial v}{\partial y}+D_{13}\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial y}\left[D_{44}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[D_{66}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+b_x=\rho_s\frac{\partial^2 u}{\partial t^2} \quad \textbf{[13]}$$

$$\frac{\partial}{\partial x}\left[D_{44}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[D_{21}\frac{\partial u}{\partial x}+D_{22}\frac{\partial v}{\partial y}+D_{23}\frac{\partial w}{\partial z}\right]+\frac{\partial}{\partial z}\left[D_{55}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+b_y=\rho_s\frac{\partial^2 v}{\partial t^2} \quad \textbf{[14]}$$

$$\frac{\partial}{\partial x}\left[D_{66}\left(\frac{\partial w}{\partial x}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[D_{55}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[D_{31}\frac{\partial u}{\partial x}+D_{32}\frac{\partial v}{\partial y}+D_{33}\frac{\partial w}{\partial z}\right]+b_z=\rho_s\frac{\partial^2 w}{\partial t^2} \quad \textbf{[15]}$$

in which, all variables have been previously defined.

2.7.2 2D Plane Strain

Similarly, the displacement based governing equations under 2D plane strain conditions can be presented as:

$$\frac{\partial}{\partial x}\left[D_{11}\frac{\partial u}{\partial x}+D_{12}\frac{\partial v}{\partial y}+D_{14}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[D_{41}\frac{\partial u}{\partial x}+D_{42}\frac{\partial v}{\partial y}+D_{44}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+b_x=\rho_s\frac{\partial^2 u}{\partial t^2} \quad \textbf{[16]}$$

$$\frac{\partial}{\partial x}\left[D_{41}\frac{\partial u}{\partial x}+D_{42}\frac{\partial v}{\partial y}+D_{44}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[D_{21}\frac{\partial u}{\partial x}+D_{22}\frac{\partial v}{\partial y}+D_{24}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+b_y=\rho_s\frac{\partial^2 v}{\partial t^2} \quad \textbf{[17]}$$

in which, all variables have been previously defined.

3 CONSTITUTIVE RELATIONSHIPS

SVSLOPE (SEISMIC) currently allows total stress analysis using a linear elastic constitutive law. More advanced material constitutive models will be added to the SVSLOPE (SEISMIC) library of constitutive models in the near future.

3.1 GENERAL

Two important decisions should be made when defining a mathematical model to describe the physical behavior of a material:

1. Selection of a constitutive model that is suitable for the type of behavior being modeled, and,
2. Entry of material behavior parameters applicable to the selected constitutive model.

3.2 ISOTROPIC LINEAR ELASTIC LAW (TOTAL STRESS)

The general incremental form of the material constitutive law, [12], for a material with an isotropic linear elastic behavior is reduced to the following format:

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \qquad \textbf{[18]}$$

where:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{12} & D_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{bmatrix}$$

$$\boldsymbol{\sigma}^T = \left\{ \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \right\}$$

$$\boldsymbol{\varepsilon}^T = \left\{ \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \right\}$$

$$D_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$D_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$D_{44} = \frac{E}{2(1+\nu)}$$

where:

E = Young's modulus
ν = Poisson ratio

For a two-dimensional plane strain analysis, ε_z, γ_{xz}, and γ_{yz} are zero. Substituting these values into [18], the out-of-plane components of stress are directly calculated as:

$$\begin{aligned} \sigma_z &= D_{12}\left(\varepsilon_x + \varepsilon_y\right) \\ \tau_{yz} &= 0 \\ \tau_{zx} &= 0 \end{aligned} \qquad \textbf{[19]}$$

Hence, [18] can be expressed only in terms of the in-plane components of stress and strain as follows:

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \qquad \textbf{[20]}$$

Where:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & D_{44} \end{bmatrix}$$

$$\boldsymbol{\sigma}^T = \left\{\sigma_x \quad \sigma_y \quad \tau_{xy}\right\}$$

$$\boldsymbol{\varepsilon}^T = \left\{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\right\}$$

The relations in [18] can alternatively be expressed in terms of the bulk and shear moduli of the material, K and G respectively. In that case, one can use the following expressions to convert the values of one set of material properties to another:

$$K = \frac{E}{3(1-2\nu)} \qquad \textbf{[21]}$$

$$G = \frac{E}{2(1+\nu)} \qquad \textbf{[22]}$$

or

$$E = \frac{9KG}{3K + G} \qquad [\ 13\ ]$$

$$\nu = \frac{3K - 2G}{2G + 6K} \qquad [\ 24\ ]$$

The linear elastic constitutive model is defined in terms of E and ν parameters in SVSLOPE (SEISMIC). The value of Poisson ratio is theoretically bounded by 0 and 0.5, respectively representing a fully compressible and a totally incompressible material. To avoid numerical difficulties arising from incompressible mechanics, SVSLOPE (SEISMIC) has limited the upper bound of Poisson's ratio to 0.499 instead of 0.5.

The P- and S-waves, introduced in Chapter 2, propagate with finite velocities in an elastic domain. The propagation velocities (not to be confused with particle velocity which is the first temporal derivative of displacement at each point of the domain) of P- and S-waves depend on the material properties of the elastic domain and can be calculated by:

$$V_P = \sqrt{\frac{K + \frac{4}{3}G}{\rho_s}} \qquad [\ 25\ ]$$

$$V_S = \sqrt{\frac{G}{\rho_s}} \qquad [\ 26\ ]$$

where:

V_P and V_S = the propagation velocities of P- and S-wave, respectively

It is obvious from [25] and [26] that $V_P > V_S$, as K and G always take positive values.

4 DISCRETIZATION AND NUMERICAL SOLUTION

The equations of motion introduced in Chapter 2 describe the propagation of (seismic) waves in the space and time domains. SVSLOPE (SEISMIC) employs a time-domain dynamic finite element method for solving these equations. To numerically solve the PDEs of motion, appropriate discretization of both space and time domains are required. This chapter explains the numerical techniques used by SVSLOPE (SEISMIC) to perform (seismic) dynamic analysis.

4.1 DISCRETE FORM OF GOVERNING EQUATIONS

SVSLOPE (SEISMIC) uses 3-node triangular elements for the discretization of the space domain in 2D problems and 4-node tetrahedral elements for the discretization of the 3D space. Using the Galerkin method (see Zienkiewicz et al. (2013) for more details) and applying the approximation of displacements corresponding to the adopted element type, the partial differential equations of motion are reduced to the following system of Ordinary Differential Equations (ODEs) in terms of the time variable, t:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \qquad [27]$$

where:

$\mathbf{M}$ = Mass matrix of the domain
$\mathbf{K}$ = Stiffness matrix of the domain
$\mathbf{F}$ = the vector of all external dynamic forces
$\mathbf{u}$ and $\ddot{\mathbf{u}}$ = the vectors of nodal displacements and nodal accelerations, respectively

For a detailed explanation of finite element approximation and the calculation of the above matrices, the user is referred to Zienkiewicz et al. (2013), Bathe (2006), Cook et al. (1989), or other finite element method references.

It should be noted that using [27] for seismic analysis preserves the kinetic energy of the system during the analysis as no mechanism for the dissipation of energy is incorporated in [27].

4.2 MASS MATRIX

The mass matrix of the model is defined by:

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^{T} \rho_s \mathbf{N} d\Omega \qquad [28]$$

where:

$\mathbf{N}$ = the matrix of finite element shape functions
Ω = the domain of analysis

In general, [28] introduces a consistent mass matrix representing a continuous mass over the domain of analysis. Alternatively, one can assume that the mass of the domain is discrete and lumped at the nodes of the finite element mesh. This assumption reduces the mass matrix to a diagonal matrix, significantly decreasing the computational cost of matrix inversion when solving the system of equations.

Several techniques exist in the literature for obtaining the lumped mass matrix for a discretized continuum (Zienkiewics et al., 2013). A common and simple though effective mass lumping technique (for domains without discontinuities) replaces the diagonal term of each row of the consistent mass matrix with the sum of its components at that row while setting all off-diagonal components to zero, i.e.:

$$\bar{m}_{ij} = \begin{cases} \sum_{j=1}^{n} m_{ij} & i = j \\ 0 & i \neq j \end{cases} \qquad [\ 29\ ]$$

where:

$\bar{m}_{ij}$ and m_{ij} = components of the lumped and consistent mass matrices, respectively

n = total number of degrees of freedom

i, j = row and column indices of mass matrix components, respectively

SVSLOPE (SEISMIC) uses a lumped mass matrix in all its (seismic) dynamic analyses.

4.3 DAMPING MATRIX

The equation of motion shown in [27] represents the motion of a system whose kinetic energy is preserved. In reality, however, the kinetic energy of a system declines over time due to the activation of internal mechanism (e.g. internal friction between soil grains) dissipating the kinetic energy of the system. To account for the energy dissipation, the average effect of all existing dissipative mechanisms is incorporated in the model by adding a viscous damping (i.e. velocity proportional) term to the equation of motion. The matrix representation of the equation of motion for a system with viscous damping can be written as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \qquad [\ 30\ ]$$

where:

$\mathbf{C}$ = Damping matrix of the domain

The damping matrix, $\mathbf{C}$, represents the average effect of all internal mechanisms that dissipate the kinetic energy of the system. Various assumptions can be made about the structure of the damping matrix based upon the behavior and micro-structure of the material. SVSLOPE (SEISMIC) uses a common classic format for the damping matrix known as the Rayleigh damping model.

Rayleigh damping model assumes that the damping matrix of a domain can be defined as a linear combination of its mass and stiffness matrices:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \qquad [31]$$

where:

α and β = coefficients of mass- and stiffness-proportional damping, respectively

Coefficients, α and β, respectively have units of s^{-1} and s.

The relationship between the damping ratios and natural frequencies of a system damped by a Rayleigh-type damping is given by a set of quadratic expressions in terms of the natural frequencies as follows (Kramer, 1995):

$$\xi_n = \frac{1}{2}\left(\alpha / \omega_n + \beta\omega_n \right) \qquad [32]$$

where:

ω_n = the natural frequency corresponding to the *n*th vibration mode of the system (a.k.a. the *n*th natural frequency of the system)

ξ_n = the damping ratio of the *n*th vibration mode of the system

Hence, to determine the Rayleigh coefficients, α and β, one can use [32] for two different natural frequencies of the system and their corresponding damping ratios to determine the two unknowns. In practice, it is common to assume that both modes have the same damping ratio. Therefore, for a given damping ratio, ξ, corresponding to two different modes of the system with natural frequencies, $\omega_1 = \omega$ and $\omega_2 = \lambda\omega$, Rayleigh coefficients can be directly calculated as:

$$\alpha = \frac{2\xi\lambda\omega}{1+\lambda} \quad \text{and} \quad \beta = \frac{2\xi}{\omega(1+\lambda)} \qquad [33]$$

where:

λ = the ratio of the two natural frequencies of the system

SVSLOPE (SEISMIC) uses user-defined values for the damping ratio and the minimum and maximum response frequencies of the model to determine Rayleigh damping coefficients from [31].

4.4 TIME DOMAIN DISCRETIZATION AND NEWMARK SCHEME

The equation of motion, [30], represents a set of ODEs with respect to time and should be integrated over the time domain to yield a set of algebraic equations that can be solved for nodal displacements of the system. Once displacements are calculated, a numerical differentiation scheme is required to obtain velocities and accelerations form the displacements of the system.

Numerical integration/differentiation over the time domain requires the time domain to be discretized. The effective size of the time increments is directly dictated by the physical properties of the space domain, such as mass and stiffness, and the numerical algorithm employed for carrying out time integration/differentiation over the time domain.

When an explicit numerical scheme (e.g. Central Difference Method) is employed, the solution is conditionally stable. In this case, the time increment used in dynamic analysis must satisfy the Courant-Friedrichs-Lewy (CFL) condition (Courant et al. 1967) which limits the maximum size of the time increment to a critical value defined as:

$$\Delta t_{cr} = L_{\min} / V_P \qquad [34]$$

where:

Δt_{cr} = the critical time increment (the maximum admissible value for a time step)

$L_{\min}$ = the smallest distance between any two nodes of the finite element mesh

V_P = velocity of P-wave in the domain

Satisfaction of the CFL condition is a requirement for the stability of the explicit numerical scheme.

When an unconditionally stable implicit scheme (e.g. Constant-average-acceleration Newmark Method) is used, the satisfaction of the CFL condition is no longer a requirement for the stability of the numerical scheme. It is recommended, however, to use time increments close to the value of the critical time increment [34] in order to preserve the accuracy of the results (Bathe, 2006). SVSLOPE (SEISMIC) generates analysis time arrays according to the sampling frequency of the input signals (defined by the user) and observing the requirements for an accurate dynamic solution.

The Newmark family of methods (Newmark, 1959) has been implemented in SVSLOPE (SEISMIC) for performing numerical integration/differentiation over the time domain. The values of acceleration and velocity at the *(n*+1*)*th discrete time step, $t_{n+1} = t_n + \Delta t$, can respectively be calculated, according to the Newmark method, as:

$$\begin{aligned} \ddot{\mathbf{u}}_{n+1} &= a_0\left(\mathbf{u}_{n+1} - \mathbf{u}_n\right) - a_1\dot{\mathbf{u}}_n - a_2\ddot{\mathbf{u}}_n \\ \dot{\mathbf{u}}_{n+1} &= \dot{\mathbf{u}}_n + a_3\ddot{\mathbf{u}}_n + a_4\ddot{\mathbf{u}}_{n+1} \end{aligned} \qquad [35]$$

where:

$$a_0 = \frac{1}{\alpha_N \Delta t^2} \quad , \quad a_1 = \frac{1}{\alpha_N \Delta t} \quad , \quad a_2 = \frac{1}{2\alpha_N} - 1$$

$$a_3 = \Delta t\left(1 - \delta_N\right) \quad , \quad a_4 = \delta_N \Delta t$$

with Newmark parameters: $\delta_N \geq 0.50$ and $\alpha_N \geq 0.25\left(0.5 + \delta_N\right)^2$

Substituting [35] into the equation of motion, [30], at time $t = t_{n+1}$, the nodal displacement vector of the model is obtained by solving the following system of algebraic equations for the displacement vector, $\mathbf{u}_{n+1}$:

$$\left(\mathbf{K} + a_0\mathbf{M} + a_5\mathbf{C}\right)\mathbf{u}_{n+1} = \mathbf{F}_{n+1} + \mathbf{M}\left(a_0\mathbf{u}_n + a_1\dot{\mathbf{u}}_n + a_2\ddot{\mathbf{u}}_n\right) + \mathbf{C}\left(a_5\mathbf{u}_n + a_6\dot{\mathbf{u}}_n + a_7\ddot{\mathbf{u}}_n\right) \qquad [36]$$

where:

$$a_5 = \frac{\delta_N}{\alpha_N \Delta t} \quad , \quad a_6 = \frac{\delta_N}{\alpha_N} - 1 \quad , \quad a_7 = \frac{\Delta t}{2}\left(\frac{\delta_N}{\alpha_N} - 2\right)$$

SVSLOPE (SEISMIC) by default uses $\alpha_N = 0.25$ and $\delta_N = 0.50$ representing the unconditionally stable constant-average-acceleration scheme that is optimum for most problems. Newmark parameters in SVSLOPE (SEISMIC) can be modified by the user; however, it is highly recommended that the user exercises caution when modifying these parameters. Inappropriately chosen values of Newmark parameters may result in an unstable scheme. Furthermore, choosing a value of $\delta_N > 0.50$ results in an artificial numerical damping that can eliminate some important frequency contents of the dynamic solution (Newmark, 1959).

5 INITIAL CONDITIONS

Seismic analysis is an Initial Boundary Value Problem (IBVP). In addition to the boundary conditions of the system at any time $t>0$, solution of an IBVP requires the initial state of the system (i.e., the state of the system at $t=0$). The initial conditions for a seismic analysis in SVSLOPE (SEISMIC) are defined in the form of initial stresses.

5.1 INITIAL STRESSES

In SVSLOPE (SEISMIC), the model is assumed initially to be in static equilibrium under the applied forces, such as gravity.

Let the initial state of stress in a model under a static equilibrium with all applied forces be denoted by $\boldsymbol{\sigma}_0$. The general form of the constitutive model presented in [11] can be re-written, without loss of generality, to account for the initial state of stress as follows:

$$\boldsymbol{\sigma} = \mathrm{S}(\boldsymbol{\varepsilon}, t) + \boldsymbol{\sigma}_0 \qquad [\ 37\ ]$$

where:

$\mathbf{S}(\varepsilon,t)$ = the change in the stress state after the initial state (i.e. during seismic analysis)

Hence, the deformations of the initial state can be discarded, and its stresses can be imported as initial stresses for seismic analysis in the constitutive law. Generally, $\mathbf{S}(\varepsilon,t)$, can also be a function of the initial state of stress, $\boldsymbol{\sigma}_0$. However, for a linear elastic constitutive law, $\mathbf{S}(\varepsilon,t)$ is independent of $\boldsymbol{\sigma}_0$.

It is imperative that any time-dependent boundary condition defined by the user be consistent, at time $t=0$, with the above assumption of initial conditions. This means that the initial stresses should be computed considering the boundary conditions of the dynamic model at t=0. In case of an inconsistent boundary condition at time $t=0$, SVSLOPE (SEISMIC) gives priority to the initial condition by overriding the boundary condition at time $t=0$ with the initial condition and applying appropriate corrections to avoid numerical instability.

6 BOUNDARY CONDITIONS

This section outlines different types of boundary conditions that can be defined for a dynamic analysis in SVSLOPE (SEISMIC).

6.1 DYNAMIC FORCES

The definition of a dynamic force in SVSLOPE (SEISMIC) involves determining the location of the boundary on which the force applies, the magnitude of the force, and its time dependency. Time dependency is a function that defines the variation of the force magnitude over time. SVSLOPE (SEISMIC) uses a separation of variables approach to implement dynamic forces in its finite element formulation. Therefore, a general dynamic force, $\mathbf{F}$, acting at point, $\mathbf{x}$, at time, t, can be defined as:

$$\mathbf{F}(\mathbf{x},t) = \overline{\mathbf{F}}(\mathbf{x})T(t) \qquad [38]$$

where:

$\mathbf{x}$ = the location vector of the point on the boundary at which the force is applied
$\overline{\mathbf{F}}$ = a vector that defines the spatial distribution of the dynamic force
T = a scalar function that defines the variation in the magnitude of the dynamic force over time

Definition of a dynamic force in SVSLOPE (SEISMIC) requires the user to define the boundary on which the dynamic force is applied, the magnitude of the force, and the distribution of the force over the boundary (i.e., defining $\overline{\mathbf{F}}(\mathbf{x})$) as well as assigning a time dependency to the force (i.e., defining $T(t)$).

6.2 CONSTRAINTS

Dynamic boundary conditions in SVSLOPE (SEISMIC) can be explained under two major categories: First, the boundary conditions that do not prevent wave reflection into the domain. Second, the non-reflecting boundary conditions that absorb the energy of the outgoing waves and prevent their reflection into the domain.

6.2.1 Free, fixed and motion boundary conditions

SVSLOPE (SEISMIC) allows the use of free and fixed boundaries in each direction of the global coordinate system. In addition to that, the user can prescribe motion on a boundary in the *x*-, *y*-, and *z*-directions to represent seismic excitation of a boundary. Prescribed motions can be defined as time-dependent displacements, velocities, or accelerations.

Defining any of the above-mentioned types of constraint on a boundary will result in the reflection of seismic waves from the boundary into the domain of analysis. While such reflection may be of interest in some cases (e.g. interaction with a rigid base), it may not be desirable in many other cases as the reflected wave can distort the solution within the domain.

Generally, a free boundary condition is suggested to be used on a boundary that represents a free surface, such as the ground surface or the surface of a slope. In this case, the reflection of the waves from the boundary is consistent with the actual behavior of seismic waves interacting with a free surface, and therefore is desirable.

Fixed boundary conditions can be used to represent a layer that is infinitely rigid compared to the domain of analysis. The reflected waves are also desirable in this case as the amount of energy absorbed by the rigid layer is negligible in comparison with the amount of energy reflected into the domain from the rigid boundary.

When an interacting rigid layer transmits motion to the domain, motion boundary conditions can be used to represent the reflection of the waves and transmission of motions at the same time. A motion boundary condition can be defined independently in each direction on a boundary. In SVSLOPE (SEISMIC), defining a motion boundary condition must be followed by assigning a pre-defined motion (displacement, velocity, or acceleration) to the direction in which the motion boundary condition is defined. For example, if a motion boundary condition is defined in the *x*-direction on a boundary, a pre-defined motion must be assigned to that boundary to define the time variation of boundary displacements, velocities, or accelerations in the *x*-direction.

When wave reflection from the boundaries is not desirable, the user should choose non-reflecting boundary conditions as explained in section 6.2.2 of this manual.

6.2.2 Non-reflecting boundary conditions

It is common in geotechnical analysis to represent a large or an infinite domain with a truncated finite domain and apply appropriate boundary conditions on the truncation boundaries to resemble the far field behavior. Defining such boundary conditions can be challenging for both static and dynamic analysis. In seismic analysis, the major challenge arising from domain truncation is the elimination of unphysical wave reflections from the boundaries of the truncated domain. The unphysical reflected waves will interact with the waves propagating within the domain and distort the results of analysis.

To overcome this limitation, the boundaries of the model should be chosen far enough such that, the reflected waves do not propagate back to the domain of interest during the analysis. This approach, however, becomes prohibitively expensive in many cases as the propagation velocity of seismic waves is significant in most geotechnical materials.

Alternatively, one can enforce appropriate boundary conditions at the truncation boundary to completely absorb the kinetic energy of the outgoing waves and prevent any reflection from the boundaries. This type of boundary condition is referred to as *non-reflecting* boundary condition.

Non-reflecting boundary conditions in SVSLOPE (SEISMIC) have been implemented based on the theory developed by Lysmer and Kuhlemeyer (1969) and extended by Joyner and Chen (1975). Lysmer and Kuhlemeyer suggested that enforcing the following set of boundary stresses can provide a perfect absorption of the outgoing waves at the boundary:

$$\begin{aligned} \sigma_n &= \rho_s V_P \dot{u}_n \\ \sigma_t &= \rho_s V_S \dot{u}_t \end{aligned} \qquad \textbf{[39]}$$

where:

subscripts *n* and *t* denote the components in the directions normal and tangential to the boundary, respectively

The values of ρ_s, V_P, and V_S are calculated from the material properties of the domain adjacent to the non-reflecting boundary.

Unlike the boundary conditions explained in section 6.2.1, SVSLOPE (SEISMIC) does not allow the independent application of non-reflecting boundary conditions in different directions on a boundary. When a boundary of a model is defined as non-reflecting boundary, the non-reflecting boundary condition will be enforced to that boundary in *x*-, *y*-, and *z*-directions.

If a motion is assumed to be transmitted from the far-field to the truncated domain, the damping of the outgoing waves and transmission of the far-field motion to the domain can be enforced simultaneously using a formulation similar to [39]:

$$\sigma_n = \rho_s V_P \left(\dot{u}_n - \dot{u}_n^g \right)$$
$$\sigma_t = \rho_s V_S \left(\dot{u}_t - \dot{u}_t^g \right) \qquad \textbf{[40]}$$

where:

$\dot{u}_n^g$ and $\dot{u}_t^g$ = the components of velocity of the far-field motion in the directions normal and tangential to the boundary, respectively

In this case, SVSLOPE (SEISMIC) allows the application of independent far-field motions in the directions normal and tangential to the boundary.

It should be noted that both [39] and [40] are used for enforcing a perfect absorption of outgoing P- and S-waves on the non-reflecting boundaries. This assumption is valid when the contrast between the rigidity of the truncated domain and the adjacent material is negligible. Therefore, no reflection occurs at their interface. If the domain is truncated at an interface of two materials with contrasting rigidity (e.g. the interface of a soil column and the underlying rock), it is normal to assume that the energy of the wave is only partially absorbed at the boundary. The remaining part of the wave energy will be transferred to the domain in the form of reflected waves. To account for this effect, Joyner and Chen (1975) presented a modified version of Lysmer and Kuhlemeyer's formulation, [40], which is used for modeling partially reflecting boundaries:

$$\sigma_n = \rho_s V_P \left(\dot{u}_n - 2\dot{u}_n^g \right)$$
$$\sigma_t = \rho_s V_S \left(\dot{u}_t - 2\dot{u}_t^g \right) \qquad \textbf{[41]}$$

It is important to note that the appropriate choice of boundary conditions for (seismic) dynamic analysis should be made by the user based on their engineering judgement. It is the user's responsibility to ascertain that the boundary conditions chosen for (seismic) dynamic analysis are good representatives of the actual conditions for the problem of interest.

7 REFERENCES

Achenbach, J. D. (1973). Wave Propagation in Elastic Solids. North Holland Publishing Company.

Bathe, K. J. (2006). Finite Element Procedures. Klaus-Jürgen Bathe.

Cook, R. D., Malkus, D. S., & Plesha, M. E. (1989). Concepts and Applications of Finite Element Analysis (3rd ed.). John Wiley & Sons, Inc.

Courant, R., Friedrichs, K., & Lewy, H. (1967). On the Partial Difference Equations of Mathematical Physics. IBM Journal of Research and Development 11(2): 215-234.

Harris, J. G. (2001). Linear Elastic Waves. Cambridge University Press.

Joyner, W. B., & Chen, A. T. F. (1975). Calculation of Nonlinear Ground Response in Earthquakes. Bulletin of the Seismological Society of America. 65(5): 1315-1336.

Kramer, S. L. (1995). Geotechnical Earthquake Engineering. Prentice-Hall, Inc.

Lysmer, J., & Kuhlemeyer, R. L. (1969). Finite-Dynamic Model for Infinite Media. Journal of Engineering Mechanics Division. 95: 859-877.

Newmark, N. M. (1959). A Method of Computation for Structural Dynamics. Journal of Engineering Mathematics, ASCE. 85(EM3): 67-94

Zienkiewics, O. C., Taylor, R., & Zhu, J. Z. (2013). The Finite Element Method: Its Basis and Fundamentals (7th ed.). Butterworth-Heinemann.