

PLAXIS LE

Groundwater

**1D/2D/3D SATURATED / UNSATURATED
FINITE ELEMENT GROUNDWATER SEEPAGE MODELING**

Verification Manual

**Written by:
The Bentley Systems Team**

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Bentley Systems Incorporated

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1 INTRODUCTION

The word "Verification", when used in connection with computer software can be defined as "the ability of the computer code to provide a solution consistent with the physics defined by the governing partial differential equation, PDE". There are also other factors such as initial conditions, boundary conditions, and control variables that also affect the accuracy of the code to perform as stated.

"Verification" is generally achieved by solving a series of so-called "benchmark" problems. "Benchmark" problems are problems for which there is a closed-form solution or for which the solution has become "reasonably certain" as a result of long-hand calculations that have been performed. Publication of the "benchmark" solutions in research journals or textbooks also lends credibility to the solution. There are also example problems that have been solved and published in User Manual documentation associated with other comparable software packages. While these are valuable checks to perform, it must be realized that it is possible that errors can be transferred from one's software solution to another. Consequently, care must be taken in performing the "verification" process on a particular software package. It must also be remembered there is never such a thing as complete software verification for "all" possible problems. Rather, it is an ongoing process that establishes credibility with time.

Bentley Systems takes the process of "verification" most seriously and has undertaken a wide range of steps to ensure that the Groundwater software will perform as intended by the theory of saturated-unsaturated water seepage.

The following models represent comparisons made to textbook solutions, hand calculations, and other software packages. We at Bentley Systems are dedicated to providing our clients with reliable and tested software. While the following list of example models is comprehensive, it does not reflect the entirety of models, which may be posed to the Groundwater software. It is our recommendation that water balance checking be performed on all model runs prior to presentation of results. It is also our recommendation that the modeling process move from simple to complex models with simpler models being verified through the use of hand calculations or simple spreadsheet calculations.

2 ONE-DIMENSIONAL SEEPAGE

The following examples compare the results of Groundwater against published 1D solutions presented in textbooks or journal papers. One-dimensional scenarios were entered in Groundwater through the use of a thin 1D column.

2.1 TRANSIENT-STATE

Transient or time-dependent models allow the benchmarking of time-stepping aspects of the Groundwater software.

2.1.1 Celia Infiltration Example

Celia (1990) presented an infiltration example comparing finite difference and finite element solutions. The example represents an approximate description of a field site in New Mexico. The model involved unsaturated infiltration into a column of 100cm in depth. Celia (1990) outlined the solution offered by both finite difference and finite element methods. The time-steps are varied to illustrate the possible variation in solution profiles. The resulting profiles presented by Celia are shown in Figure 1.

Project: WaterFlow
Model: celia1990_GT

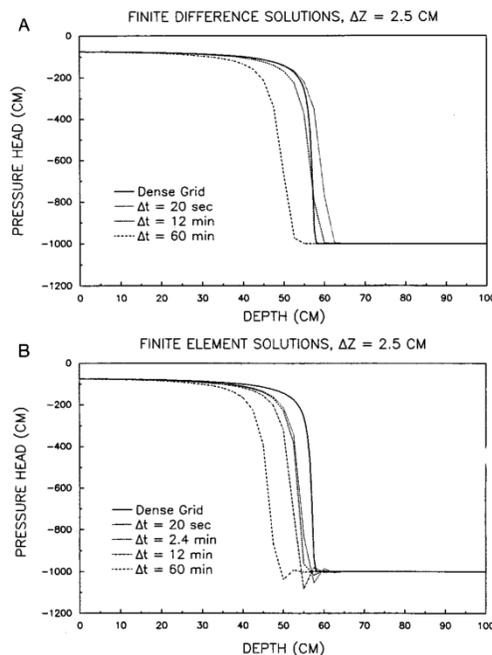


Figure 1 (a) Finite difference and (b) finite element solutions from Celia (1990). Finite difference solution using $\Delta t = 2.4$ min did not converge in nonlinear iteration.

The model was setup in the Groundwater software package. Unsaturated material property functions presented in the paper were converted from a functional to a digital representation. The results of Groundwater as compared to the finite element results presented by Celia are shown in Figure 2. A fixed mesh of 200 nodes was used in the model. Preliminary sensitivity analysis indicates that differences between the solutions can be attributed to differences in the representation of material properties.

Groundwater results indicate correct solution of the infiltration model. The results also validate the automatic time-step selection used by Groundwater in solving transient models. Numerical oscillations commonly encountered by the selection of large time-steps in finite element solvers can be minimized using Groundwater.

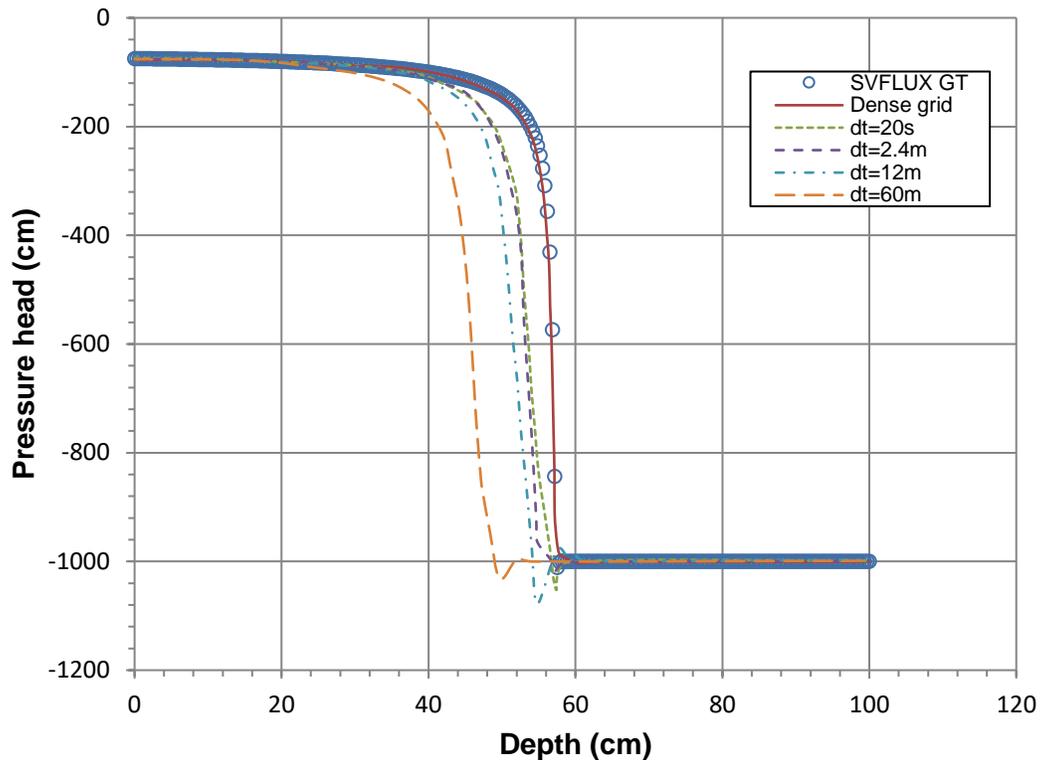


Figure 2 Difference between finite element solutions presented by Celia and the solution obtained using Groundwater

2.1.2 Evaporation – Wilson (1990)

Project: USMEP
 Model: LimitingFunction1997_SVFlux_GT, WilsonPenman1994_SVFlux_GT, EmpiricalAE_SVFlux_GT

The classic solution to the coupling of material-atmosphere equations is presented by Wilson (1990). In Wilson’s PhD thesis (1990), a column of sand was subjected to drying in a laboratory environment in which the temperature and relative humidity were controlled. Measurements of actual evaporation and the distributions of temperature along the column depth were obtained, providing several measures that can be used for the verification of the numerical model.

2.1.2.1 Model geometry and boundary conditions

A Modified Penman approach to the calculation of actual evaporation was presented in the thesis (hereafter termed the Wilson-Penman method). Wilson (1990) coded a 1D finite element package termed “Flux” in order to compare the physical results to a numerical solution. The geometry and configuration of the column may be seen in the following figure.

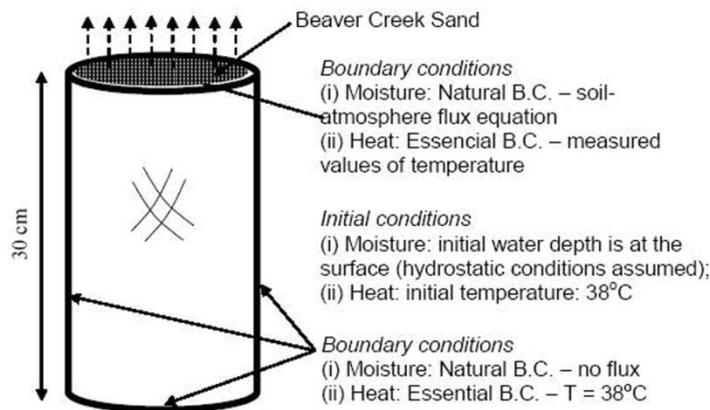


Figure 3 Numerical simulation of the drying column test (Wilson, 1990)

Three approaches are available to calculate the actual evaporation: Wilson-Penman AE (Wilson, 1994), Limiting-Function AE (Wilson et al., 1997), and Empirical AE (Wilson et al., 1997). Each approach can be simulated using uncoupled evaporative simulations using Groundwater package.

2.1.2.2 Material properties

The material properties in Wilson (1990) thesis are presented as follows. The k_{sat} value used in the numerical modeling is presented as 3×10^{-5} m/s. The unsaturated hydraulic conductivity and gravimetric water content values calculated using the Brooks and Corey estimation method are presented in Table 6.2 (p. 252). In the "FLUX" code developed by Wilson (1990), the Brooks and Corey method of representing the SWCC and unsaturated hydraulic conductivity function was used. General hydraulic properties of the Beaver Creek sand are presented in Table 4.1 (Wilson, 1990, p. 115).

In this benchmark the soil water characteristic curve (SWCC) is approximated with Fredlund and Xing (1994) approach based on the Wilson (1990) measured data. The parameters for SWCC and hydraulic conductivity are presented in Table 2, Figure 4 and Figure 5.

Table 1 Material properties used in the simulation of Wilson's evaporation benchmark

Material name	Material properties	Method and parameters	Value	unit
Beaver Creek Sand	SWCC	Sat vwc	0.405	m^3/m^3
		Fredlund and Xing		
		af	4.140	kPa
		nf	3.5759	
		mf	0.9514	
	hr	12.42	kPa	
	Hydraulic conductivity	Saturated k	2.592	m/day
		Modified Campbell Estimation		
		k min	10^{-7}	m/day
		Mcampbell p	15	

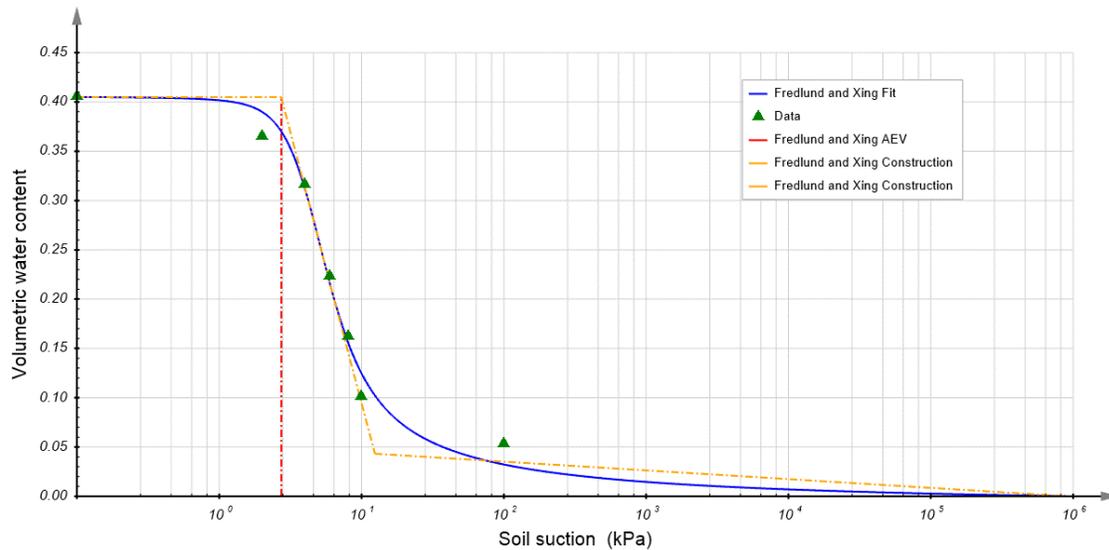


Figure 4 Soil water characteristic curve of Beaver Creek Sand used in Wilson's evaporation benchmark

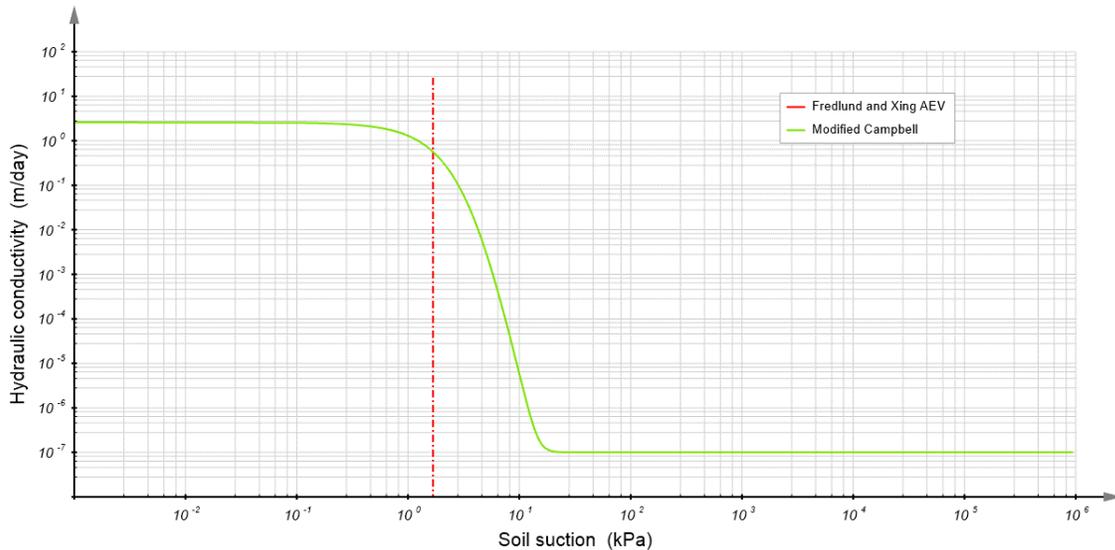


Figure 5 Hydraulic conductivity of unsaturated Beaver Creek Sand used in Wilson evaporation benchmark

2.1.2.3 Results and discussion

Exactly replicating this benchmark was technically challenging because the original code made use of an “L” parameter in the Brooks and Corey estimation in order to adjust the prediction. The use of such an “L” parameter is not currently implemented in Groundwater. Therefore the Fredlund & Xing SWCC fitting curve and the Modified Campbell hydraulic conductivity fitting curve were used and adjusted to fit the data originally published by Wilson (1990) in Table 6.2.

The experimental results obtained by Wilson (1990) were then again compared to Groundwater. The results are shown in Figure 6. It can be seen from the results that a reasonable comparison is obtained. It was found the correction number is related to material properties such as the value of k min (see Table 1). It was also found in the course of the comparison that i) the separation point between the AE and PE as well as ii) the calculated AE later on in the calculation is highly sensitive to slight variations in the representation of the SWCC and the unsaturated hydraulic conductivity curve. To improve the modeling stability, an empirical correction number in Groundwater is utilized to account for the steep suction gradient at the soil surface (see Groundwater Theory Manual for details). The suction correction factor of 1.2 in this benchmark was determined by trial and error.

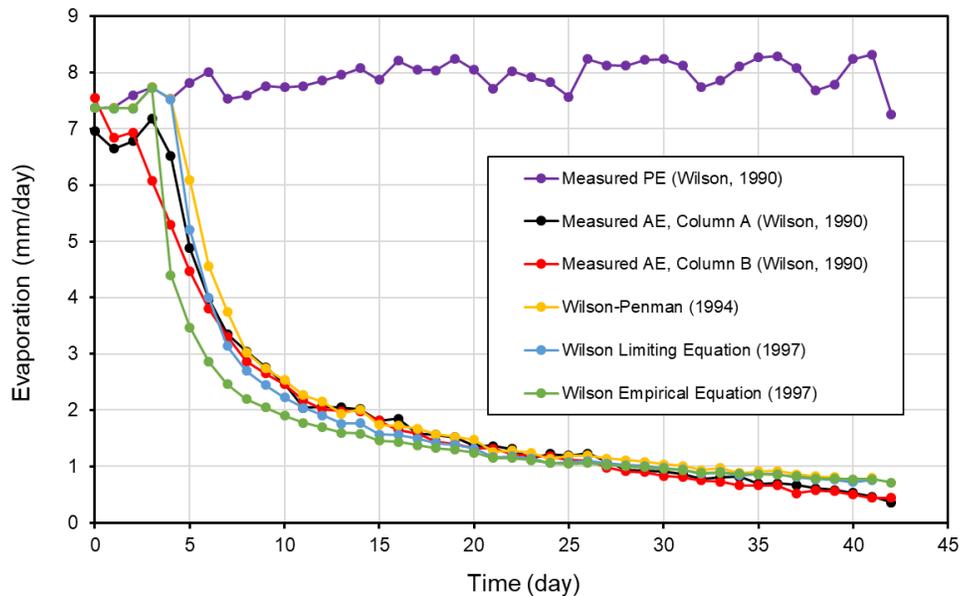


Figure 6 Comparison of evaporation simulated using Groundwater with laboratory data and numerical result by Wilson (1990)

2.1.3 Gitirana (2005) Infiltration Examples

Project: Columns

Models: InfiltrationWithRO_Gitirana2005_prec1ksat_GT; InfiltrationWithRO_Gitirana2005_prec1p5ksat_GT;
 InfiltrationWithRO_Gitirana2005_prec2ksat_GT; InfiltrationWithRO_Gitirana2005_prec4ksat_GT;
 InfiltrationWithRO_Gitirana2005_prec10ksat_GT

Gitirana (2005) presented a series of numerical models designed to test the ability of seepage software to handle cases of varying infiltration. The specific initiative involved determining the reasonableness of runoff calculations given increasing application of top boundary flux. For the series of models created each one had a varying application intensity scaled to the saturated hydraulic conductivity of the model. The models were all unsaturated and homogeneous models but each displayed the appropriate decay in actual infiltration, which would occur as the models became saturated and the amount of runoff therefore increased.

Application rates of 1x, 1.5x, 2x, 4x, and 10x saturated hydraulic conductivity were set up. The benchmarks also demonstrate the robustness of the numerical model in handling cases of increasingly high precipitation events. As the intensity of the precipitation increases, it becomes increasingly difficult to handle the increase numerically. Groundwater admirably handles intensity applications up to 10x ks_{sat}. It is therefore recommended to evaluate soil cover models in light of how the intensities compare to the ks_{sat} of the top material in the numerical model. In the series of models created a constant flux is applied to the top of the model. The model eventually saturates and runoff begins to occur. This is demonstrated in the following figures which match well with the original results presented by Gitirana (2005).

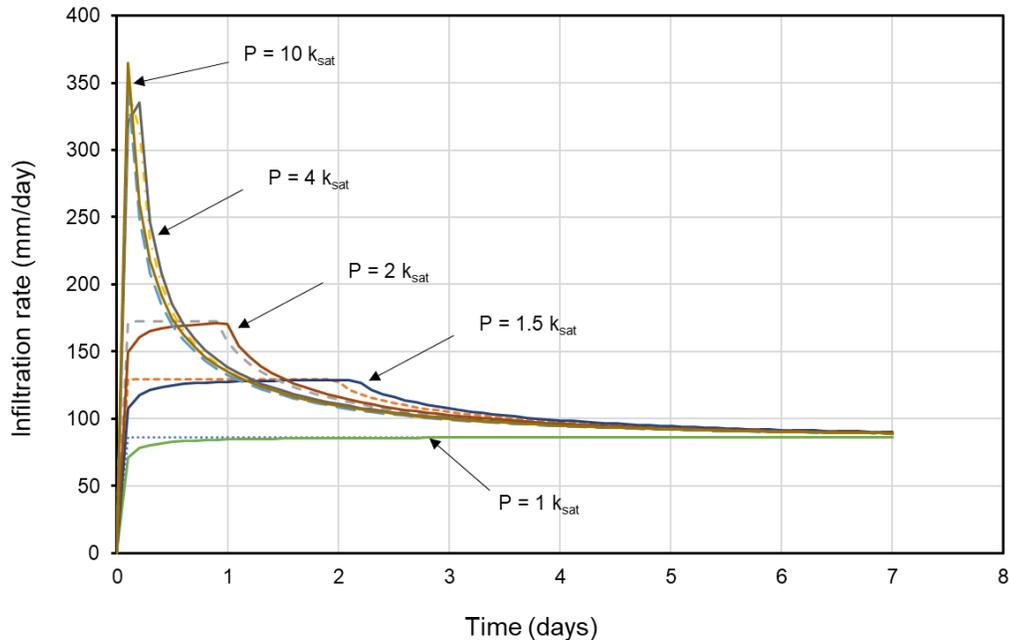


Figure 7 Graph of infiltration rate versus time

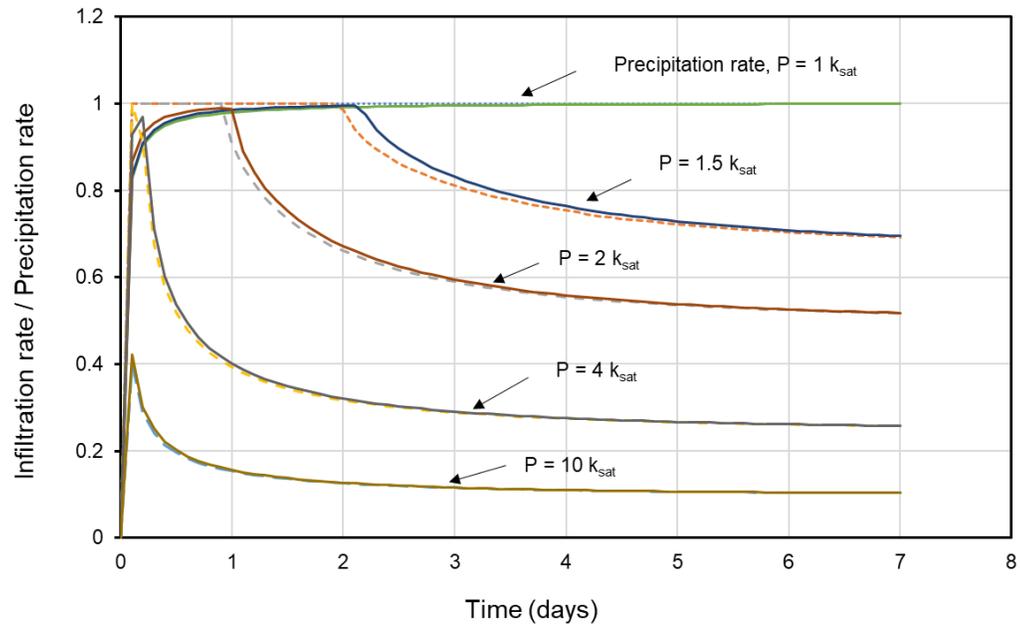


Figure 8 Graph of ratio of infiltration rate / precipitation rate versus time

It can be seen from the preceding figures that the Groundwater software performs exceptionally well in solving under extreme conditions.

3 TWO-DIMENSIONAL SEEPAGE

Various models are used to verify the validity of the solutions provided by the Groundwater software. Comparisons are made either to textbook solutions, journal-published solutions, or other software packages.

3.1 STEADY-STATE

3.1.1 2D Cutoff

Project: EarthDams
 Model: Cutoff_GT

This steady-state model is used to simulate the flow beneath a concrete gravity dam. On the left hand side of the model a reservoir is simulated by applying a constant head of 60 m while on the right side the water table is placed at the ground surface by setting a head of 40 m. All other boundaries are set to zero flow. The mesh is refined where the high gradients may occur as shown in Figure 9.

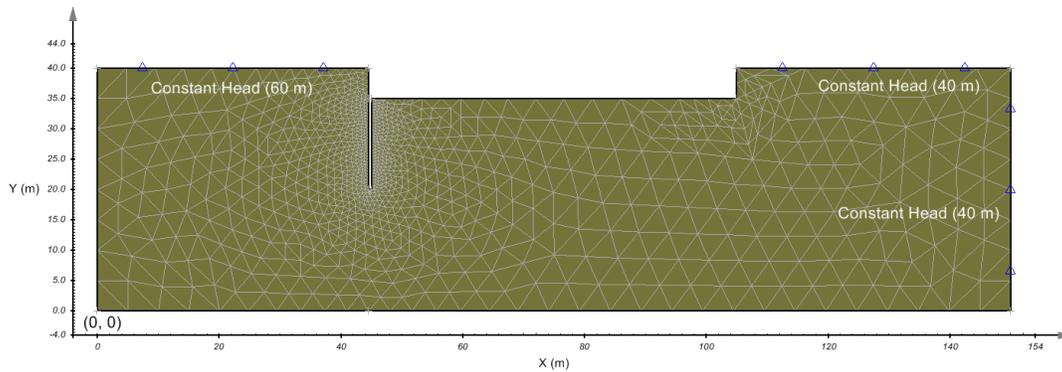


Figure 9 Mesh and boundary conditions from Groundwater

Table 2 shows the details of the material used in the model.

Table 2 Details of material properties		
Tabs	Parameters	Soil
New Material	Data Type	Saturated
Volumetric Water Content	Saturated VWC	0.33
Hydraulic Conductivity	ksat (m/s)	10 ⁻⁷

The model has been verified by using SVFLUX GE. Therefore in the present model, the results from Groundwater are compared with those from SVFLUX GE. As indicated in Figure 10, Figure 11, Figure 12 and Figure 13, the results of *h* (total head) and *uw* (pore water pressure) are in good agreement between SVFLUX GE and Groundwater. The mesh from the GE solution is also utilized in the GT solution in this example. A total of 1,039 nodes were used in the solution.

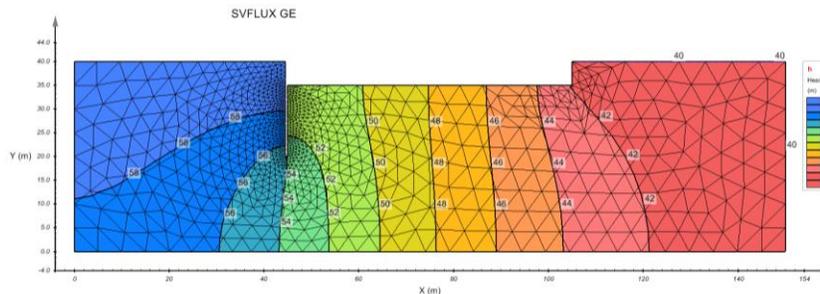


Figure 10 The result of head contours from SVFLUX GE

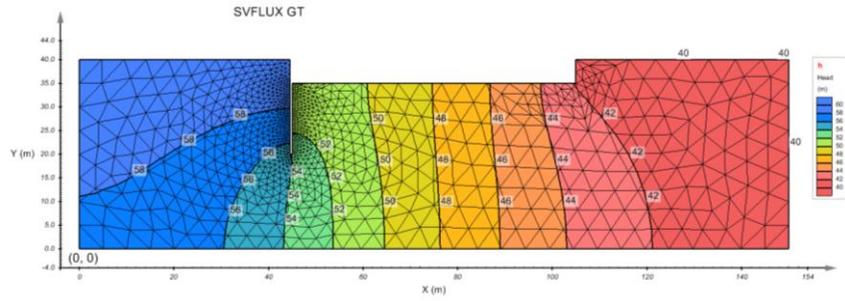


Figure 11 The result of head contours from Groundwater

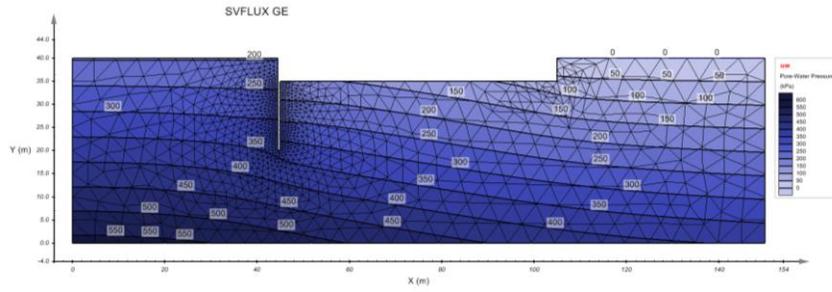


Figure 12 The result of pore water pressure contours from SVFLUX GE

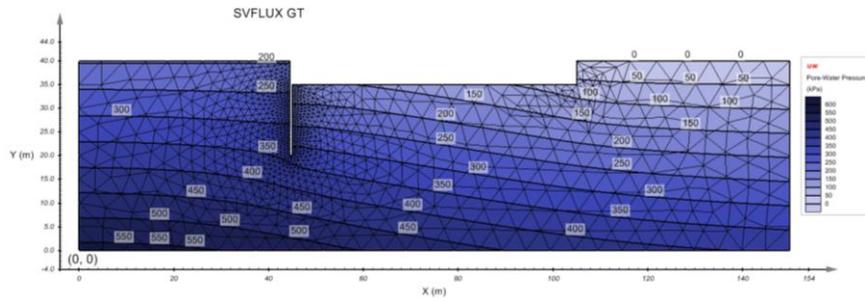


Figure 13 The result of pore water pressure contours from Groundwater

3.1.2 Dam Flow

Project: EarthDams
Model: Bowles91a_GT

This model is presented to show verification of flow through a dam cross-section using Groundwater. The left side of the dam is set as Constant Head (18 m), and the right side is viewed as Review Boundary condition, as shown in Figure 14. The other boundary conditions are viewed as "Zero Flux". The material used in the model is saturated with the saturated VWC of 0.35 and constant saturated hydraulic conductivity of 6.67×10^{-6} m/s.

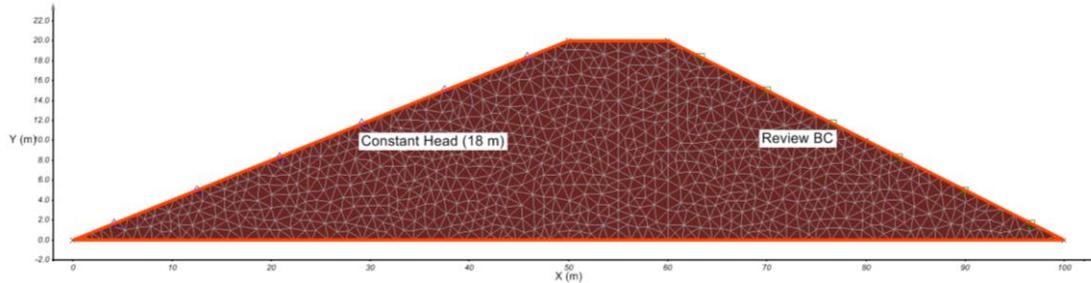


Figure 14 Geometry and boundary conditions of the model

This model is documented in Bowles (1984). The result of total head (h) from Groundwater is compared with that from Chapuis et al. (2001) in Figure 15 and Figure 16. The results are in good agreement. Figure 17 shows the contour of pore water pressure (uw) in the dam.

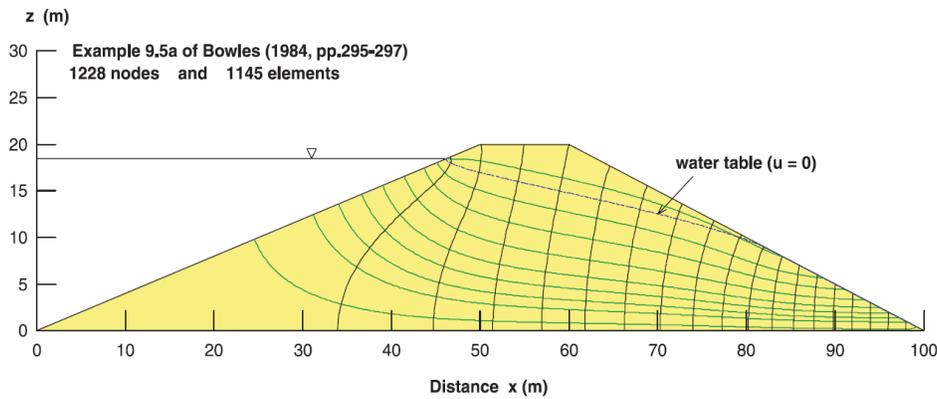


Figure 15 Total head (h) result from Chapuis et al. (2001)

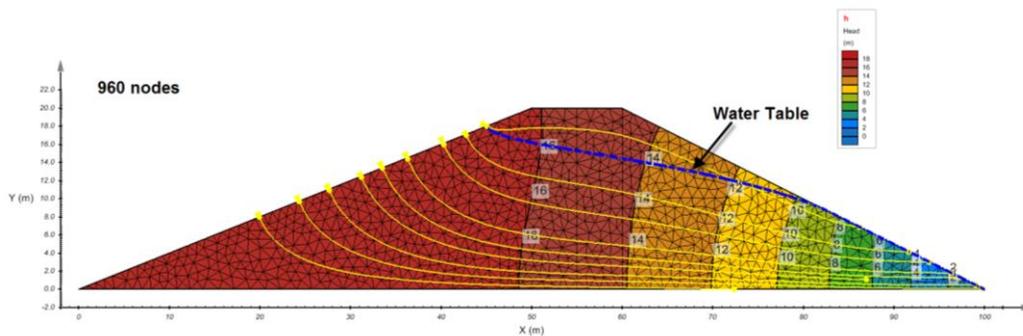


Figure 16 Total head (h) result from Groundwater

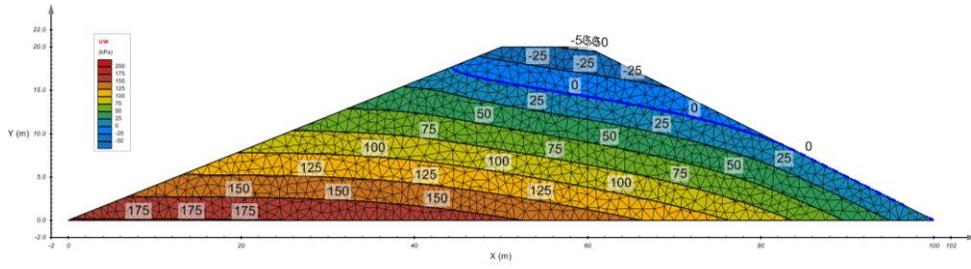


Figure 17 Pore water pressure (uw) result from Groundwater

3.1.3 Refraction Flow Example

Project: Waterflow
 Model: Crespo_GT

This model provides verification of the "refraction law" (Crespo, 1993) when water passes from one material to another. The solution can be verified using either the flow lines or equipotentials since these lines are perpendicular in the steady-state solutions when k is isotropic. A square domain of 10 m x 10 m is set up, and the boundary conditions are shown as in Figure 18.

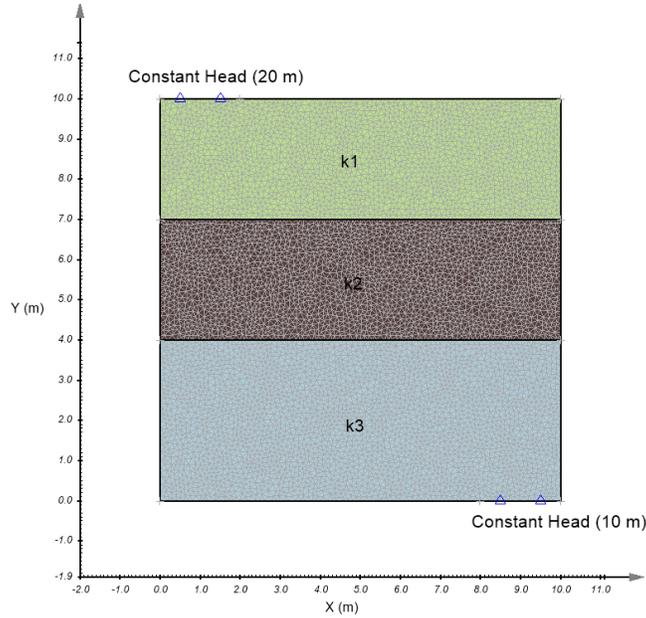


Figure 18 Simulation domain and boundary conditions settings from Groundwater (6,662 nodes)

The outer perimeter is impervious with the exception of the specified head boundary conditions. The layer thicknesses are 4 m, 3 m, and 3 m starting at the bottom. Table 3 shows the material properties for different layers in the model. The result from Groundwater is compared with that from Chapuis et al. (2001). As indicated in Figure 19 and Figure 20, it can be seen that the results from Groundwater and from the journal paper are in very good agreement.

Table 3 Details of material properties

Tabs	Parameters	K1	K2	K3
New Material	Data Type	Saturated	Saturated	Saturated
Volumetric Water Content	Saturated VWC	0.35	0.35	0.35
Hydraulic Conductivity	ksat (m/s)	0.01	0.1	0.03

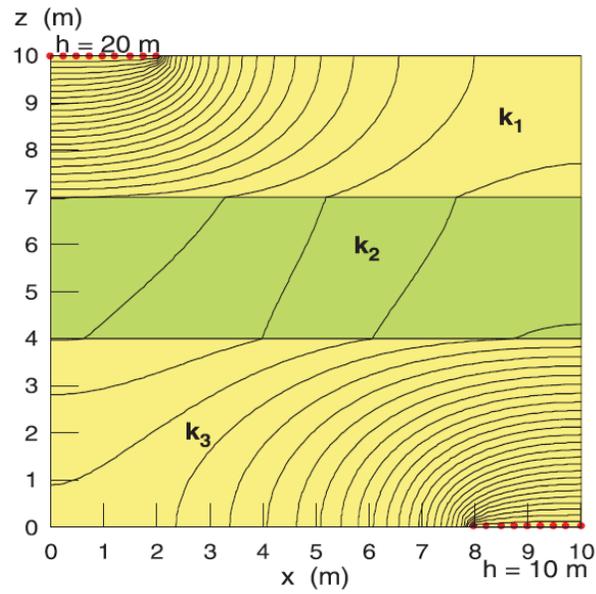


Figure 19 Results from Chapuis et al. (2001)

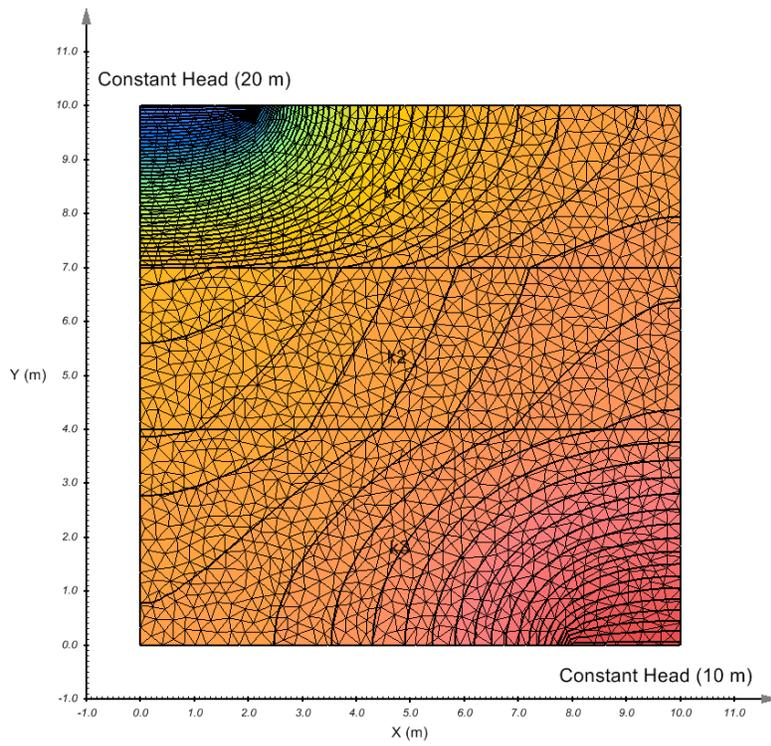


Figure 20 Verification of the refraction law using Groundwater

3.1.4 Confined Flow Under a Dam

Project: EarthDams
 Model: Example7p17_GT

This model illustrates a steady-state confined flow under a dam. The dam has two 10 m sheet piles driven partially into the granular soil layer as shown in Figure 21. On the left side of the dam, the boundary condition is set as constant pressure head (12 m), and on the right side, the boundary condition is assumed as constant pressure head (0 m). The material is viewed as saturated with the saturated volumetric water content of 0.4 and a constant saturated hydraulic conductivity of 10^{-5} m/s.

This example is taken from Holtz and Kovacs (1981). The distribution of pressure heads (h_p) at the bottom of the dam (point A through F) can be analytically calculated, and this distribution is important for the analysis of the stability of concrete gravity dams.

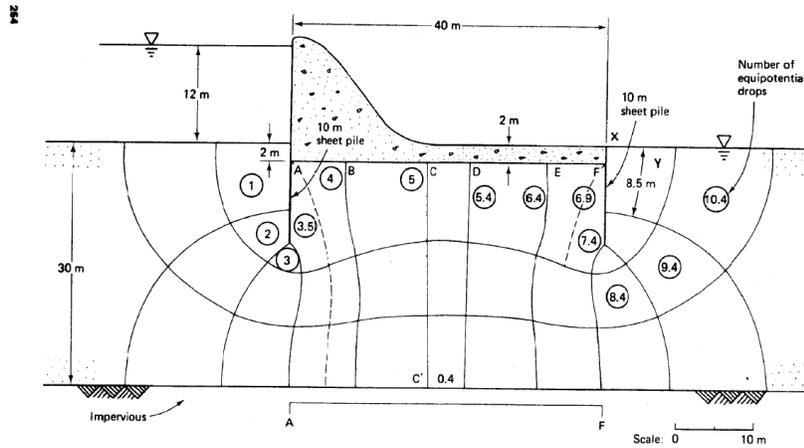


Figure 21 Description of the example model

Figure 22 represents the geometry and boundary conditions in Groundwater for simulating this example model. The mesh utilized for this problem had 2,071 nodes.

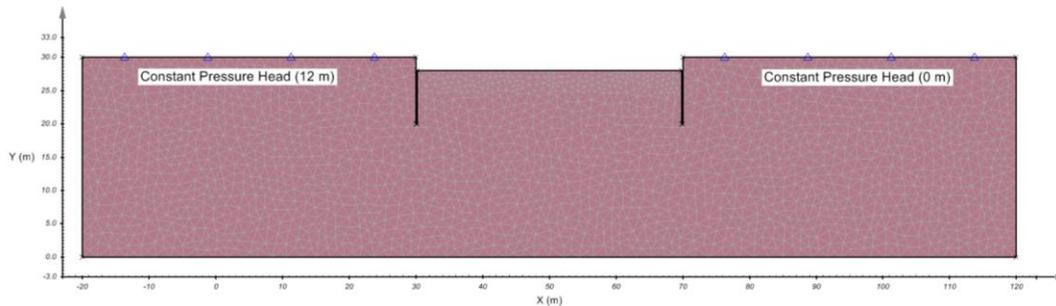


Figure 22 The geometry and boundary conditions settings in Groundwater

Figure 23 shows the contour of head (h) and several select streamlines under the dam, and the distributions of pressure heads at the bottom of the dam (from A to F) are compared between the analytical results and Groundwater in Figure 24. From the comparison, we can see that the results from the analytical calculation and Groundwater are in good agreement.

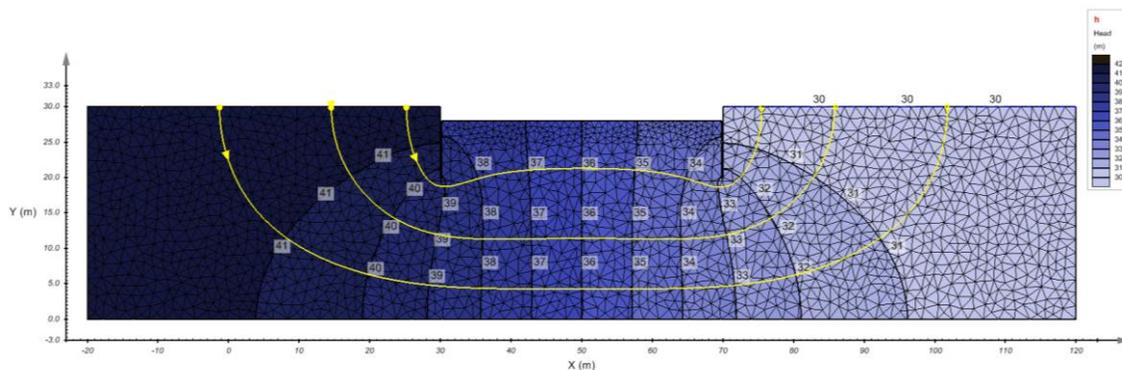


Figure 23 The contour of head (h) and select streamlines under the dam

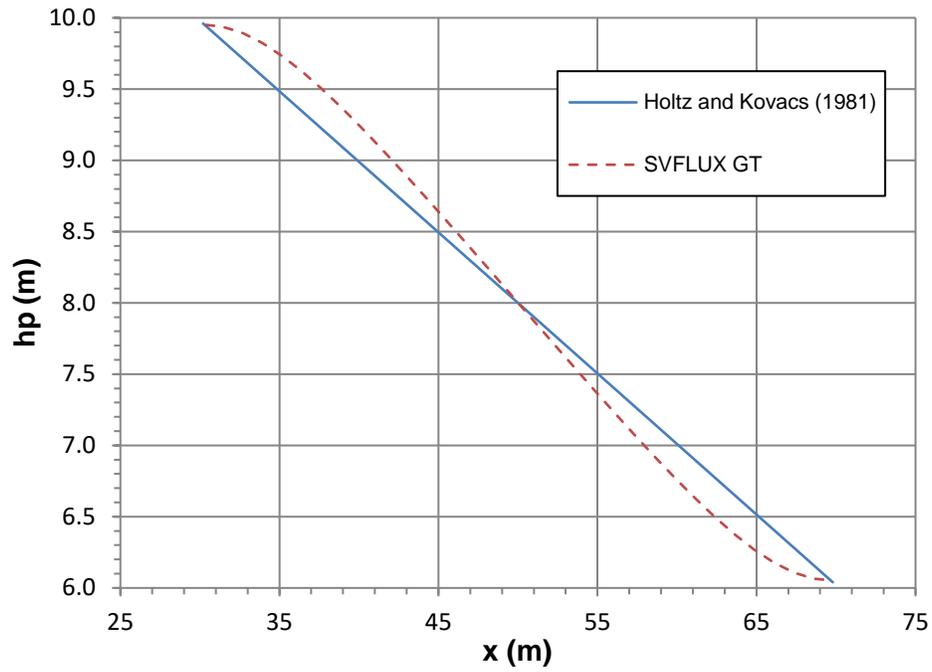


Figure 24 Comparison of the pressure head (h_p) distributions at the bottom of the dam (from A to F) between the analytical calculation by Holtz and Kovacs (1981) and Groundwater

3.1.5 Flow Around a Cylinder

Project: WaterFlow
 Model: FlowAroundACylinder_GT

Laminar flow around a cylinder is one of the most basic problems in Fluid Mechanics. The analytical solution can be obtained using the Bernoulli equation. This steady-state model simulates uniform flow around a cylinder to verify the ability of Groundwater to solve laminar flow models.

This example considers flow around a cylinder in a 2D confined space (8 m × 8 m) caused by a difference in head between the left and right sides of the model. Only half of the geometry is simulated in Groundwater as shown in Figure 25, because of the symmetry of the problem. This problem was meshed with 2,004 nodes. Constant Head boundary conditions are set as 4 m and 6 m at the left and right sides, respectively, and this difference of heads causes a flow throughout space in the negative x-direction. Meanwhile, the top, bottom and cylinder sides are viewed as "Zero Flux" boundary conditions. The radius of the cylinder is 1 m. The material is viewed as saturated with the saturated VWC of 0.4 and constant saturated hydraulic conductivity of 10⁻⁵ m/s.

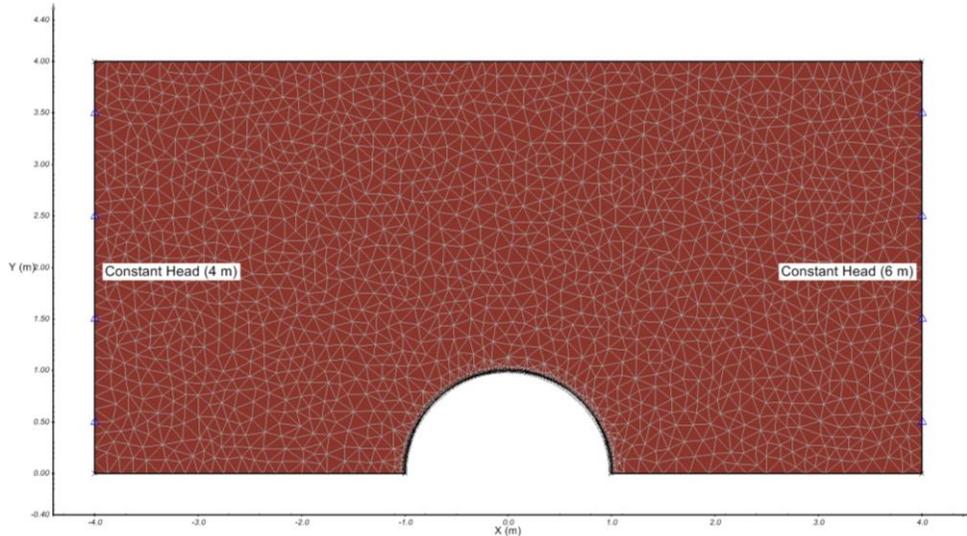


Figure 25 Geometry and boundary conditions of the model of Flow Around a Cylinder in Groundwater

According to Streeter (1966), the total head (*h*) at any point (*r*, *θ*) in the radial coordinates can be obtained from the equation [1]:

$$h = U \left(r + \frac{a^2}{r} \right) \cos \theta + 5 \tag{1}$$

Where

$$U = \frac{h_1 - h_2}{L} \tag{2}$$

- a* is the radius of the cylinder (1 m);
- h*₁ is the head at the left boundary (4 m);
- h*₂ is the head at the right boundary (6 m);
- L* is the length of the domain (8 m).

Figure 26 shows the contour of total head (*h*) from Groundwater.

Figure 27 shows the comparison of *h* along *y* = 2 m between Groundwater and analytical results from Streeter (1966). It can be seen that the analytical and Groundwater results are in good agreement.

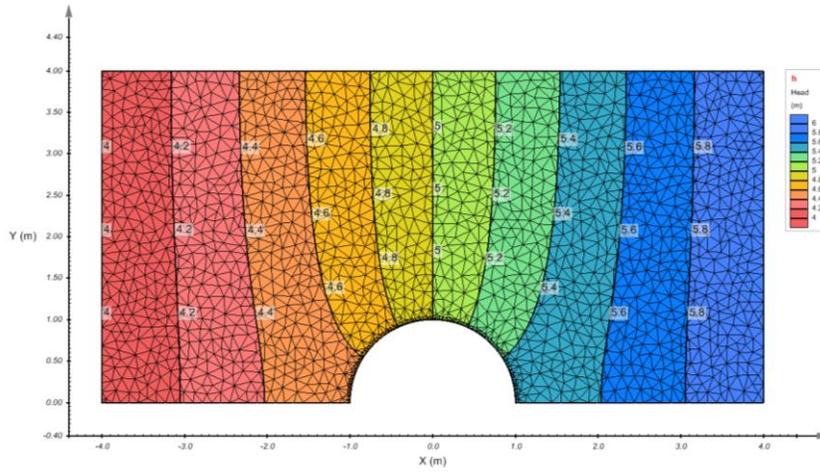


Figure 26 Contour of total head (h) from Groundwater

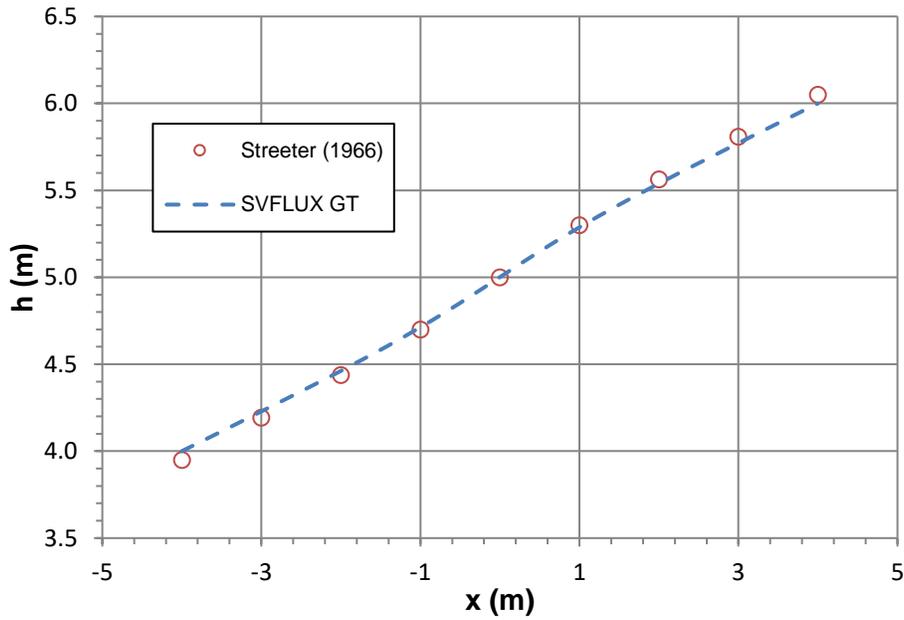


Figure 27 Comparison of total head (h) along y = 2 m between Groundwater and analytical solution from Streeter (1966)

3.1.6 The Interaction of Lakes and Ground Water

Project: RegionalFlow

Model: MODFLOW_8_GT; MODFLOW_9_GT; MODFLOW_10_GT; MODFLOW_11_GT; MODFLOW_13_GT; MODFLOW_14_GT; MODFLOW_15_GT; MODFLOW_16_GT; MODFLOW_17_GT; MODFLOW_18_GT;

The interaction between surface waters and the surrounding groundwater regime has important implications for the effective protection and management of groundwater resources. Winter (1976) used numerical simulation to examine ground-water flow systems near lakes and the general principles of the interaction of lakes and ground water. In this verification report, several select models in the paper by Winter (1976) are simulated using Groundwater. The results are compared with those from the paper and are proven to be in good agreement. These examples verify the ability of Groundwater in the hydrogeological field with simulating anisotropic flows.

The original verification report was concerned with a two-dimensional, steady-state, non-homogeneous, anisotropic one-lake system. This system is composed of a lake (shallow or deep), sediments, different water tables (low or high) at upslope or downslope sides, extensive or partial aquifers at the base or middle. The approximate dimensions of features in this one-lake system are shown in Figure 28. The system can be thought of as a ground-water reservoir that has a datum of 100 feet (bottom of the ground-water reservoir). The values of head are in feet relative to that datum, thus the lake surface has an altitude of 230 feet, the higher water table at the upslope is 300 feet, and the water table at the downslope end is 170 feet.

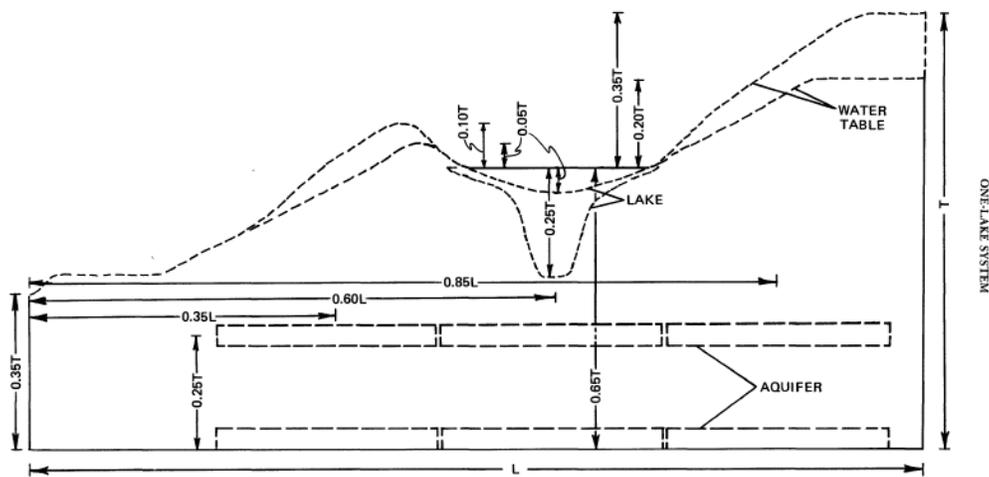


Figure 28 Dimensions of the one-lake system.

The boundary conditions at where the water table locates are set as zero pressure head. The two vertical boundaries and the base of the system are viewed as zero normal flux boundary conditions. As suggested by Winter (1976), the hydraulic conductivities assigned to lake sediments are as low as can be assigned with the software at hand, and the sediments are purposely not extended to the shore line. Lake water is simulated as a separate region and by assigning very high hydraulic conductivity.

Besides different geometry settings, two key parameters in the ground-water flow models are examined: the ratio of horizontal to vertical hydraulic conductivity (K_h/K_v) and the ratio of the hydraulic conductivity of aquifers to that of the surrounding till (K_{aq}/K_t). Based on different geometries and material properties, ten models are simulated using Groundwater and can be summarized in Table 4.

Table 4 Summary of model settings

Number	Lake depth	Sediments	Upslope	Downslope	k_h/k_v	k_{aq}/k_t	Full Aquifer	Partial Aquifer
1	shallow		High	High	1000			
2	shallow	Yes	High	High	1000			
3	shallow	Yes	High	Low	1000			
4	shallow	Yes	Low	Low	1000			
5	shallow	Yes	High	High	1000	1000	Base	
6	shallow	Yes	High	High	1000	100	Middle	
7	shallow	Yes	Low	Low	1000	1000		Base, Up
8	shallow	Yes	Low	Low	1000	1000		Base, Beneath

9	shallow	Yes	Low	Low	1000	1000		Base, Down
10	Deep	Yes	High	High	1000	100	Base	

Results

One of the most important phenomena in understanding the interaction of lakes and ground water is the stagnation point. This is a point in the flow field at which vectors of flow are equal in opposite directions and therefore cancel. If a stagnation point exists that has a head greater than that of the lake level, a continuous ground-water divide exists beneath the lake making it impossible for water to move against the hydraulic gradient from the lake to the ground-water system. Table 5 summarizes the stagnation point results from Groundwater and from the simulations by Winter (1976). Some differences exist, but overall, Groundwater successfully predicts the existence of the stagnation points.

The following figures show the comparisons of head contours between Groundwater and Winter (1976). The results from Groundwater are in good agreement with those from Winter (1976). From both quantitative and qualitative comparisons, these examples further verify the ability of Groundwater in the field of hydrogeological simulations where anisotropic flows are considered.

Table 5 Summary of stagnation point results from Groundwater and Winter (1976)

Model	Interaction	Stagnation point from Groundwater (m)	Stagnation point from Winter(1976) (m)
1	No	1.8	1.8
2	No	2.9	4.2
3	No	0.9	1.4
4	No	0.5	0.9
5	Yes	---	---
6	Yes	---	---
7	No	0.8	1.3
8	No	2.4	1.8
9	Yes	---	---
10	Yes	---	---

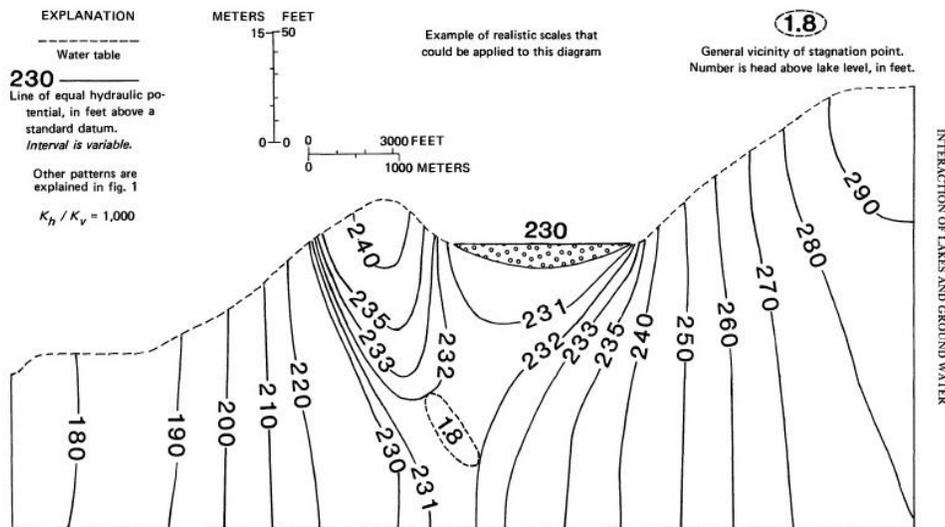


FIGURE 8.—Hydrologic section showing distribution of hydraulic head in the ground water reservoir of a one-lake system that does not contain aquifers or lake sediments.

Figure 29 Head contours of model 1 from Winter (1976).

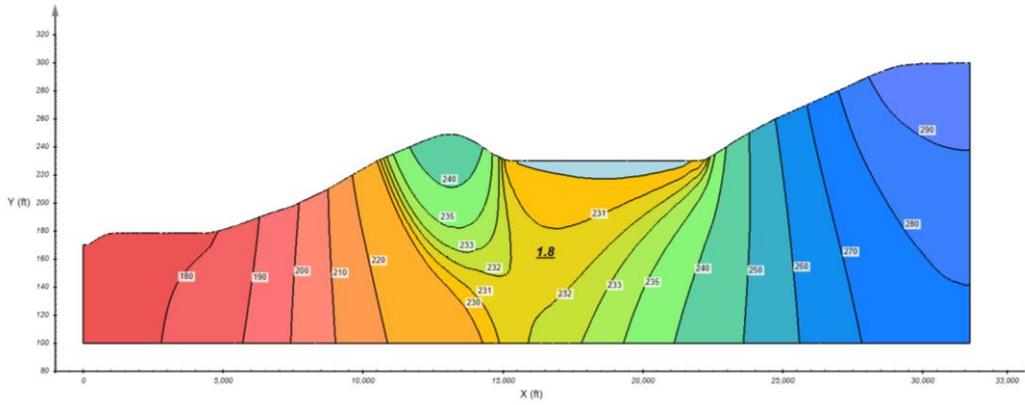


Figure 30 Head contours of model 1 from Groundwater.

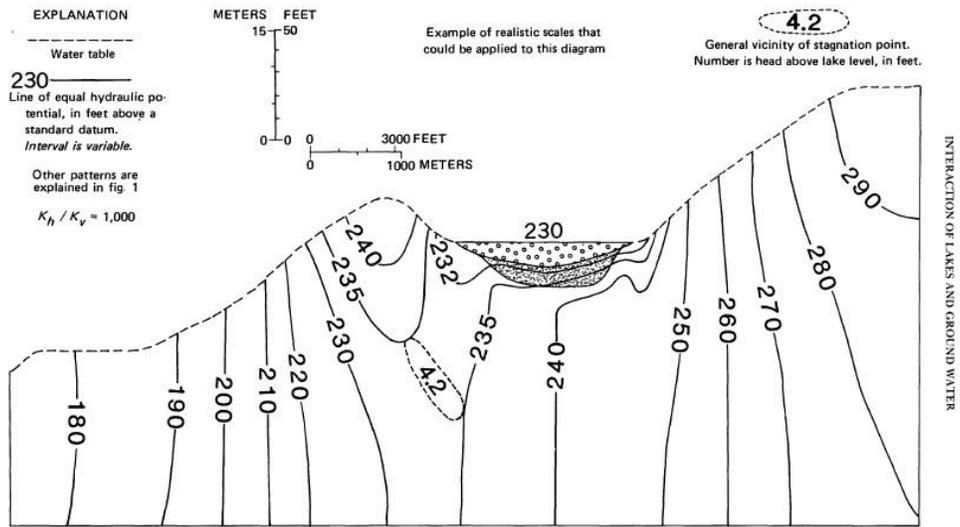


FIGURE 9.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that contains lake sediments but does not contain aquifers.

Figure 31 Head contours of model 2 from Winter (1976).

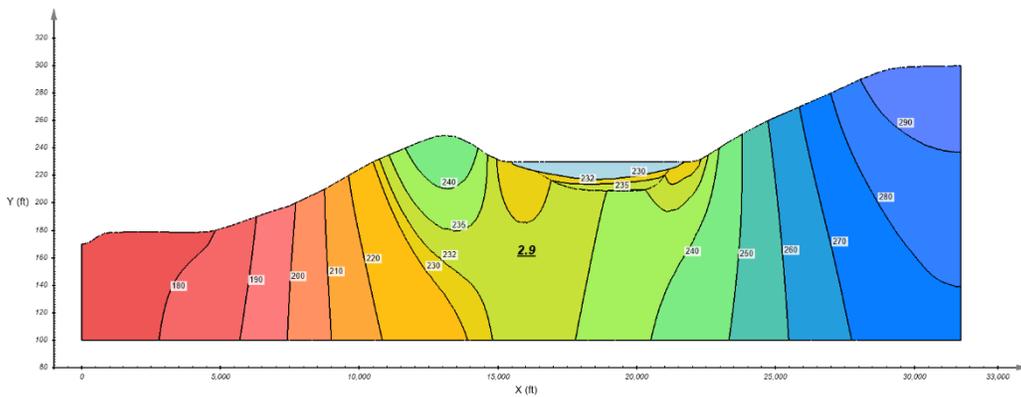


Figure 32 Head contours of model 2 from Groundwater.

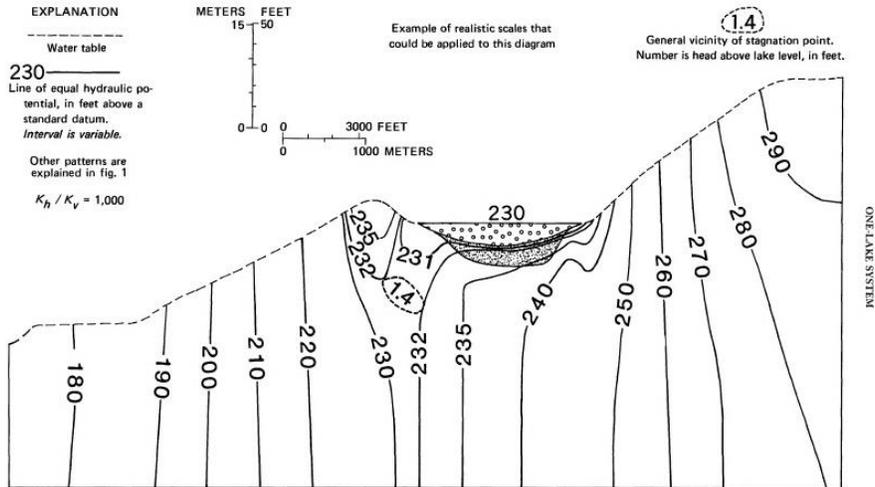


FIGURE 10.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that has a low water-table mound on the downslope side of the lake and does not contain aquifers.

Figure 33 Head contours of model 3 from Winter (1976).

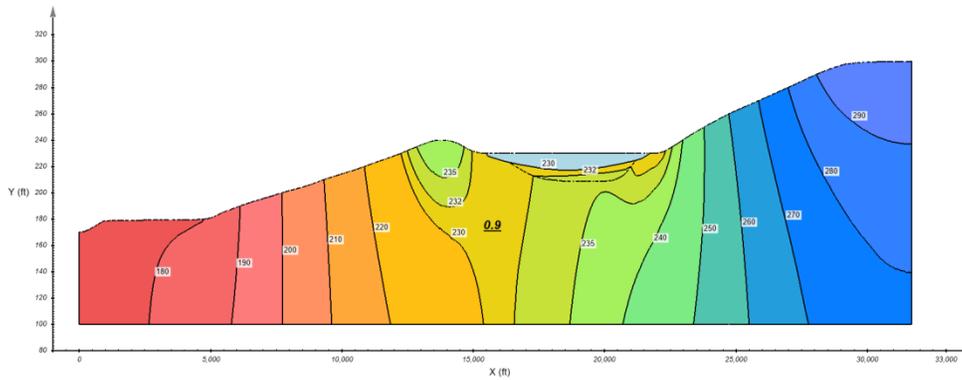


Figure 34 Head contours of model 3 from Groundwater.

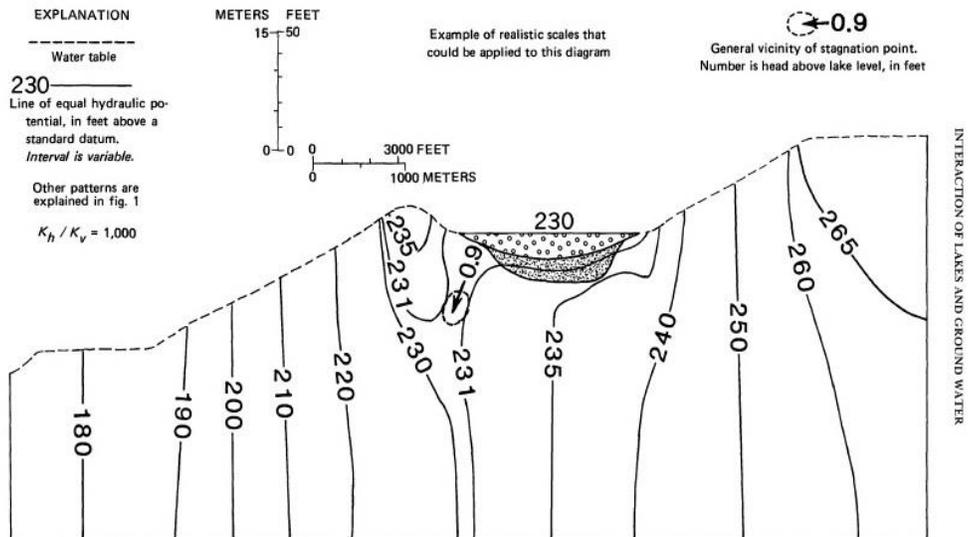


FIGURE 11.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that has low water-table mounds on both sides of the lake and does not contain aquifers.

Figure 35 Head contours of model 4 from Winter (1976).

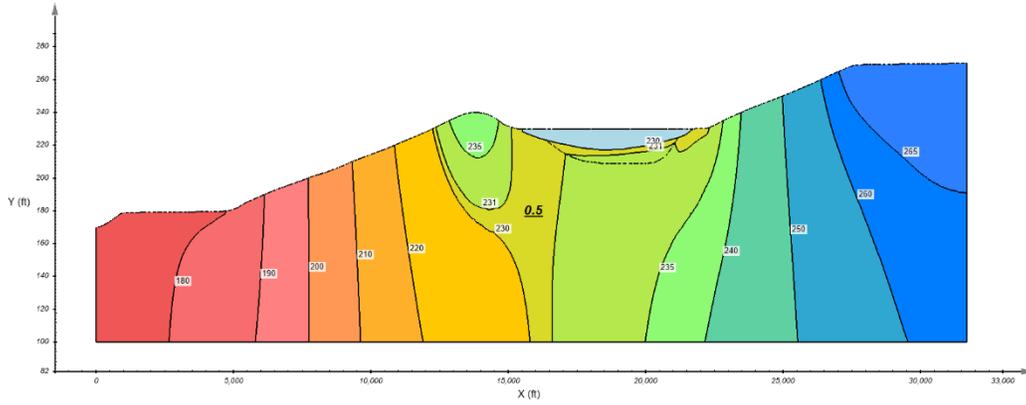


Figure 36 Head contours of model 4 from Groundwater.

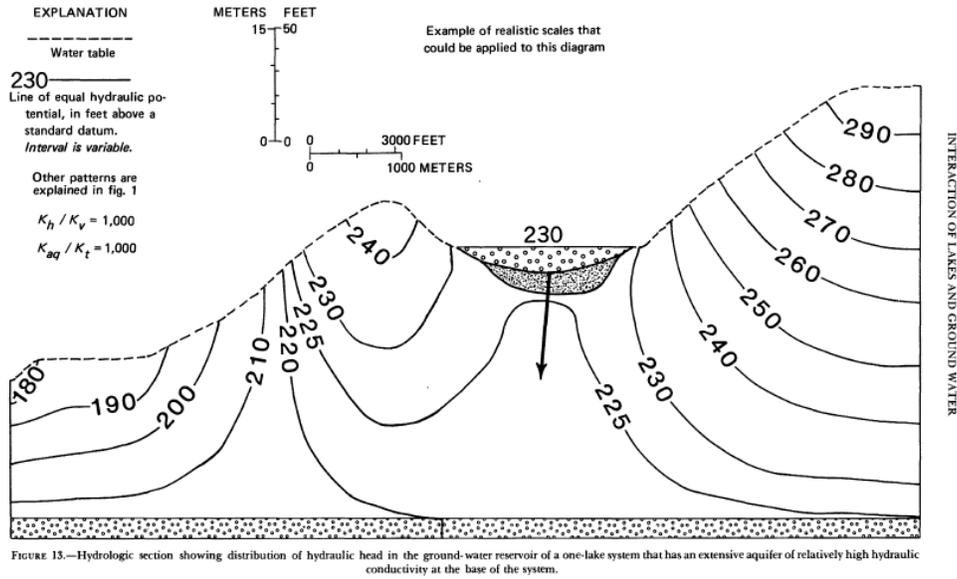


FIGURE 13.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that has an extensive aquifer of relatively high hydraulic conductivity at the base of the system.

Figure 37 Head contours of model 5 from Winter (1976).

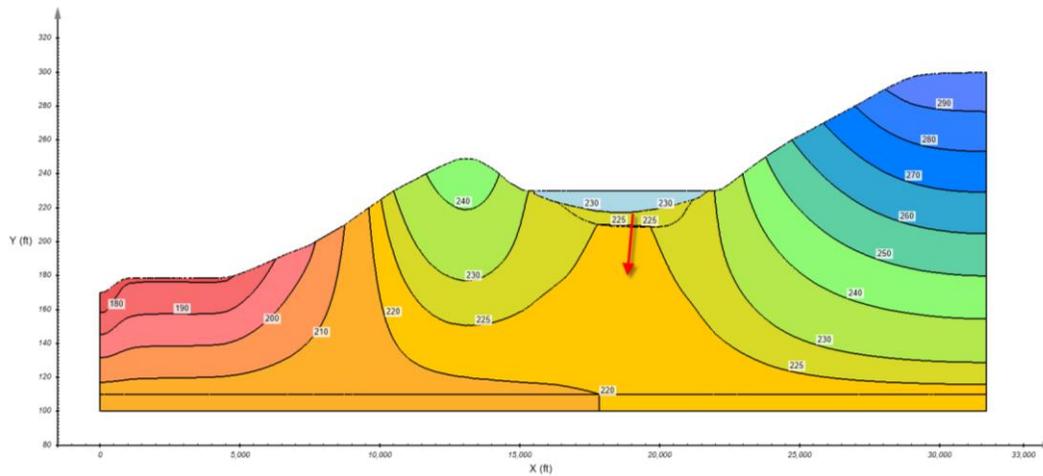


Figure 38 Head contours of model 5 from Groundwater.

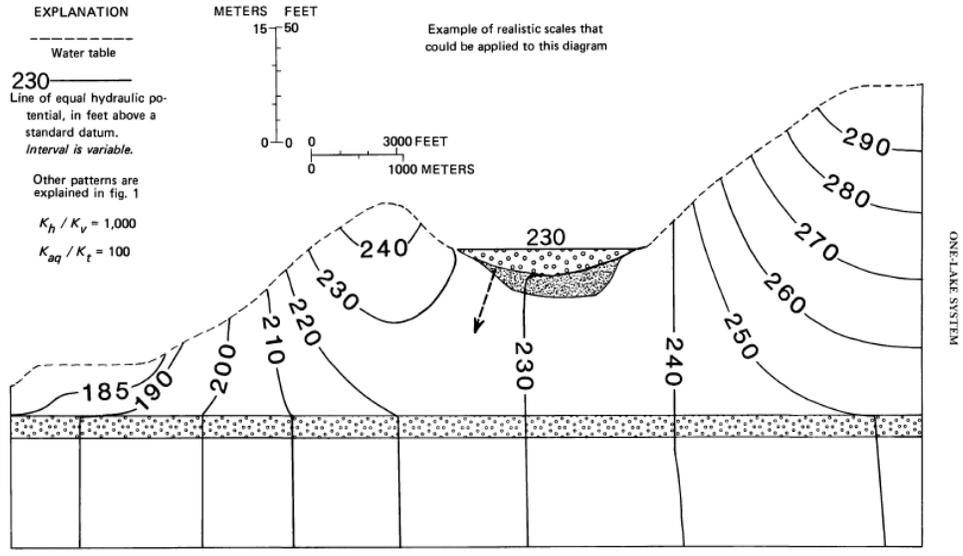


FIGURE 14.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that has an extensive aquifer of relatively low hydraulic conductivity in the middle of the system.

Figure 39 Head contours of model 6 from Winter (1976).

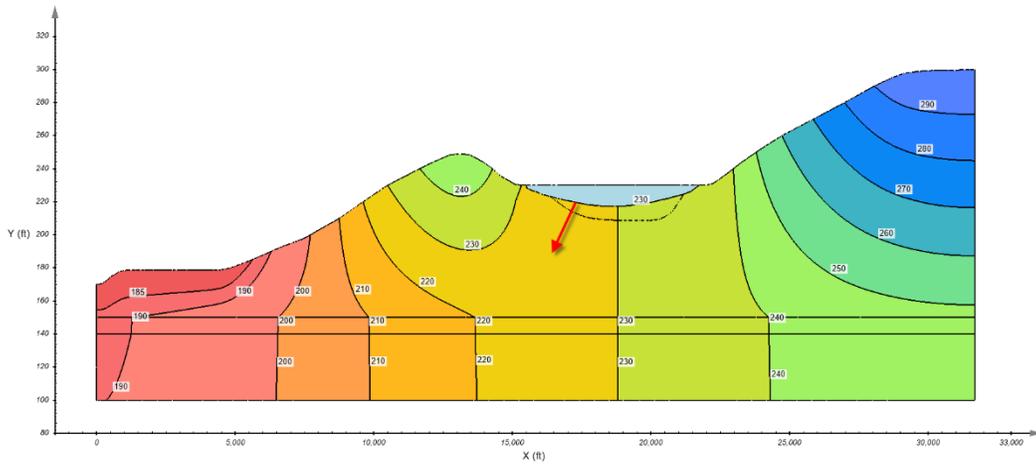


Figure 40 Head contours of model 6 from Groundwater.

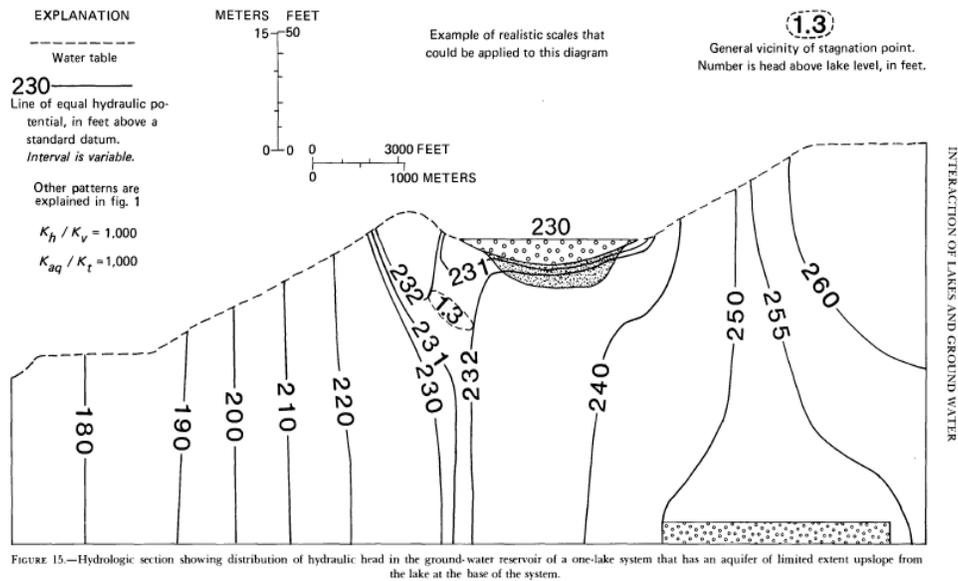


Figure 41 Head contours of model 7 from Winter (1976).

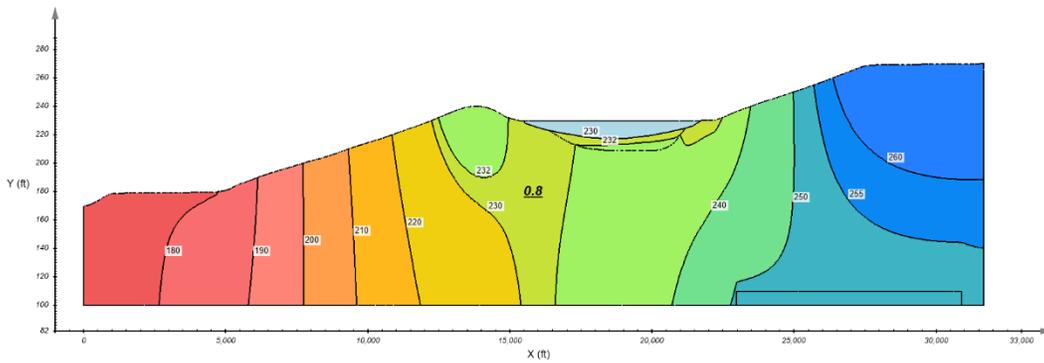


Figure 42 Head contours of model 7 from Groundwater.

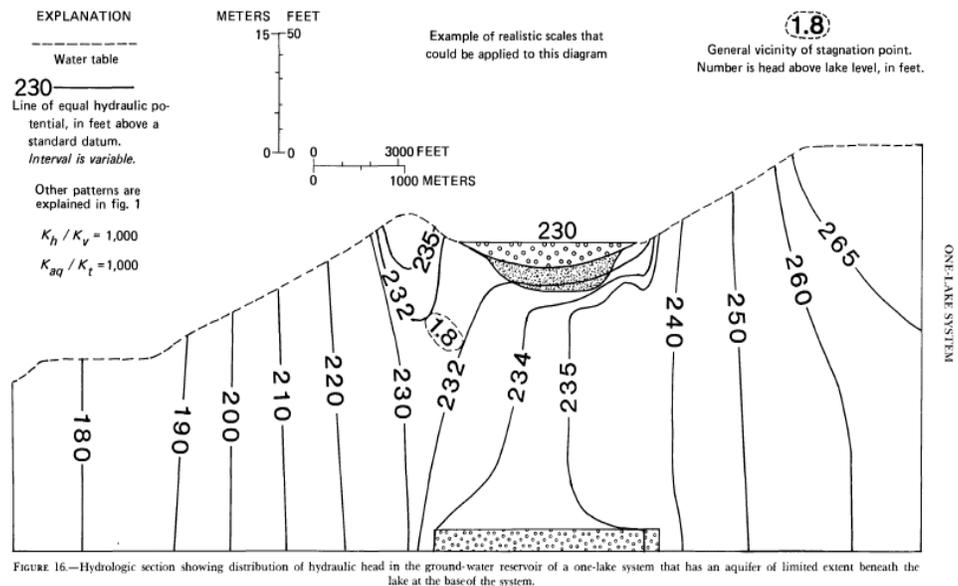


Figure 43 Head contours of model 8 from Winter (1976).

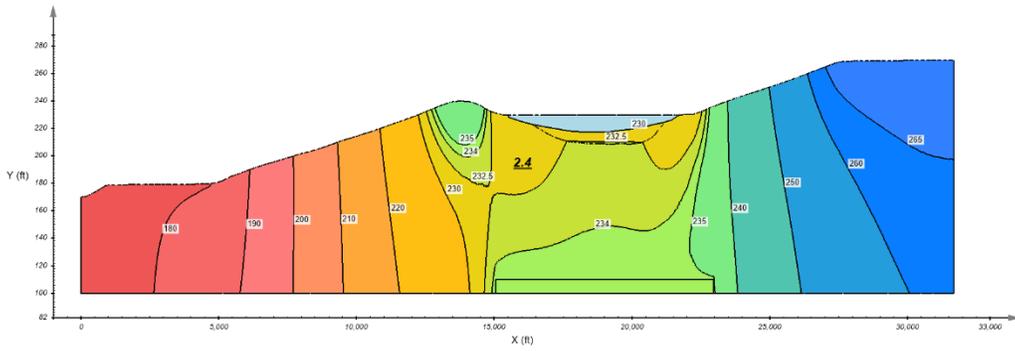


Figure 44 Head contours of model 8 from Groundwater.

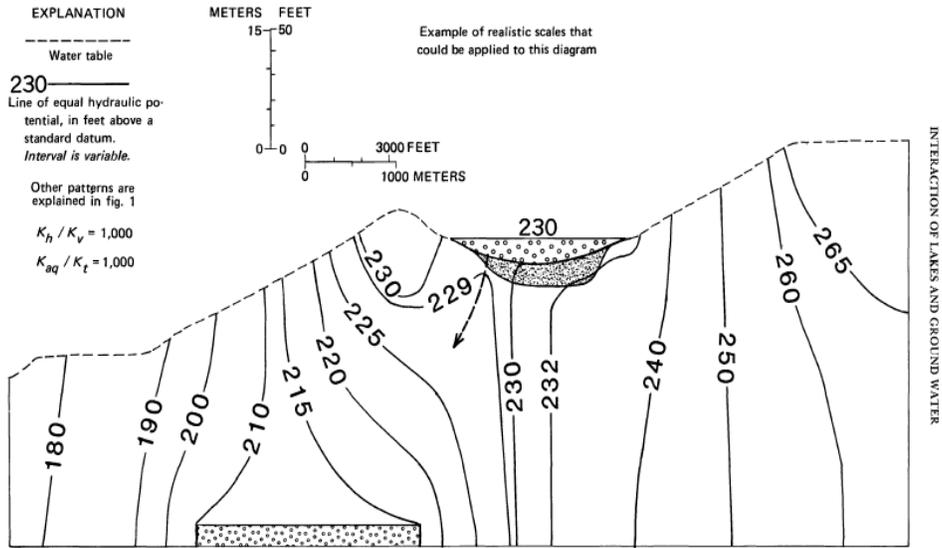


FIGURE 17.—Hydrologic section showing distribution of hydraulic head in the ground-water reservoir of a one-lake system that has an aquifer of limited extent downslope from the lake at the base of the system.

Figure 45 Head contours of model 9 from Winter (1976).

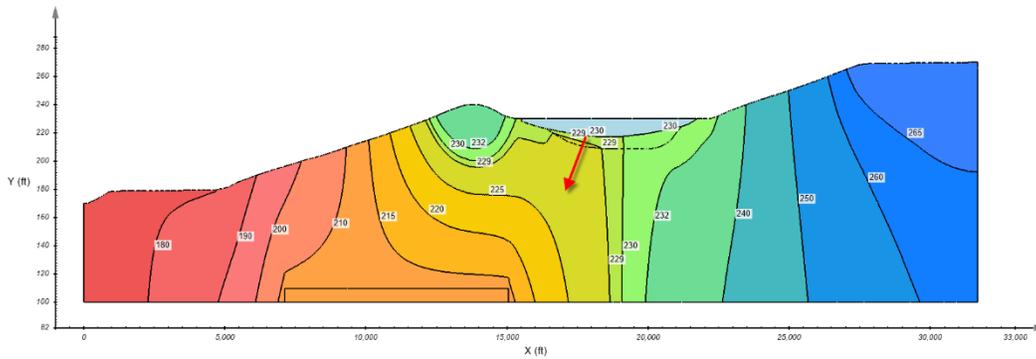


Figure 46 Head contours of model 9 from Groundwater.

3.1.7 Dam with Unconfined Groundwater Flow (Muskat)

Project: EarthDams

Model: Muskat_h2_GT; Muskat_h4_GT; Muskat_h6_GT; Muskat_h8_GT; Muskat_h10_GT

This model is used to verify the groundwater flow capabilities of Groundwater. This verification model considers a vertical cross-section of an unconfined groundwater flow system in a homogeneous earth dam underlain by an impervious base, and a free-surface and a seepage face appear atop the flow region as shown in Figure 49. The results from the simulation of Groundwater are compared to the numerical and analytical results from the journal paper by Lee and Leap (1997).

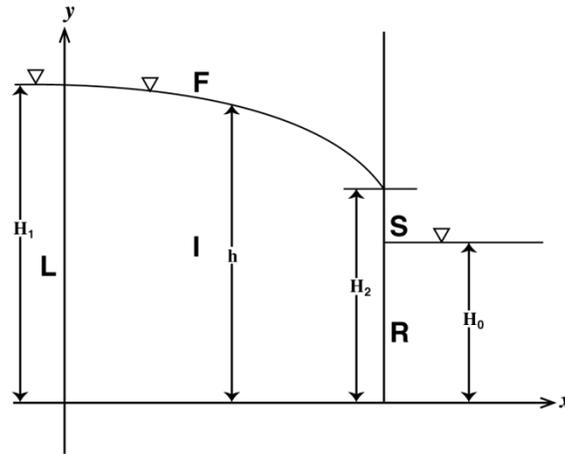


Figure 49 Physical domain of the model.

Figure 50 indicates the mesh and boundary conditions of the model. The meshes for different models vary from 2,000 nodes to 10,000 nodes. The geometry of the simulation domain is a 20 m X 20 m square, and an initial water table line is set at the top surface. The boundary condition on the left side is assigned as Head Constant = 20 m. On the right side, the boundary condition from $y = 0$ m to $y = H_0$ is assigned as Head Constant = H_0 , and from $y = H_0$ to $y = 20$ m as Review Boundary Condition to determine the length of the seepage surface. The mesh is locally refined around the central part of the right side. Five different values of H_0 (2 m, 4 m, 6 m, 8 m and 10 m) are tested.

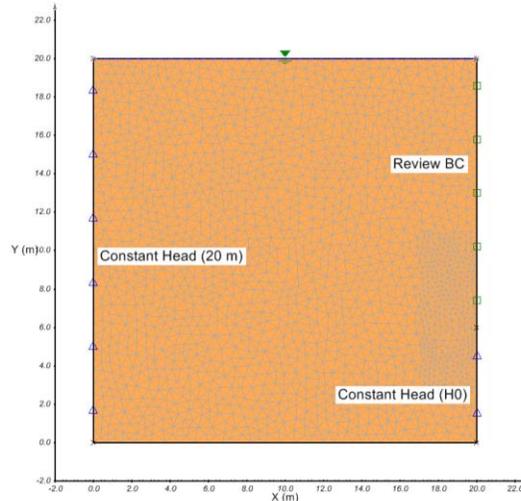


Figure 50 The mesh and boundary conditions of the model

Table 6 shows the details of the materials properties used in the model.

Table 6 Details of material properties		
Tags	Parameters	Soil
New Material	Data Type	Unsaturated
Volumetric Water Content	Saturated VWC	0.402
	SWCC	Fredlund and Xing Fit
Hydraulic Conductivity	ksat (m/s)	3.5×10^{-4}
	Unsaturated Hydraulic Conductivity	Modified Campbell Estimation

The results of the model and the results from Lee and Leap (1997) are summarized in Table 7. From the comparisons it can be seen that the results from Groundwater are close to the simulation results from Lee and Leap (1997). Some differences exist because the length of the seepage surface is very sensitive to the mesh density along the boundary near the exit point and in the area nearby. Overall, the results can be viewed as comparable with those in the paper.

Table 7 Results and comparisons

H_1 (m)	H_0 (m)	S (m) ^[1] (Lee, 1997)	S (m) (analytical)	S (m) (Groundwater)	Error ^[2]
20	2	5.7	5.3	5.70	0.0%
20	4	4.0	3.7	4.10	2.5%
20	6	2.7	2.4	2.70	0.0%
20	8	1.7	1.5	1.60	5.9%
20	10	0.9	0.8	0.80	11.1%

^[1]Simulation results from Lee and Leap, 1997.

^[2]Compare with the simulation results from Lee and Leap, 1997.

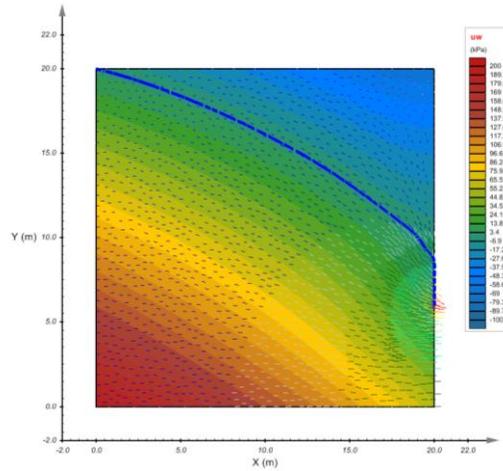


Figure 51 Pore water pressure result and the final location of the water table line

3.1.8 Seepage In Layered Hill Slope

Project: USMEP_Textbook
 Model: LayeredHillSlopeSeepage_GT

This model considers the problem of the seepage in a layered hill slope. Figure 52 indicates the geometry and boundary conditions of the model. A constant normal flux (2.1×10^{-4} m/s) is applied to the top side of the slope, and the initial water table is located at $y = 0.3$ m. The boundary condition of the slope face was set as Review Boundary Condition. The physical description of the model can also be found in the book by Fredlund and Rahardjo (1993).

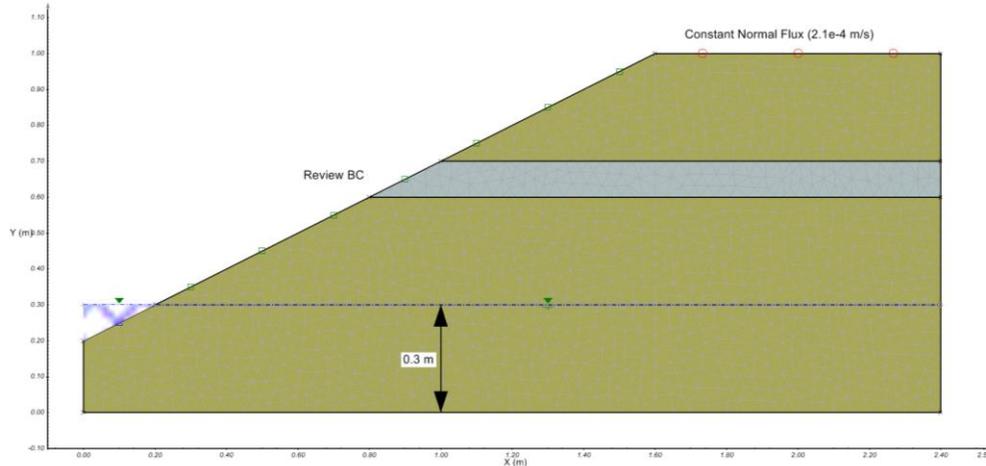


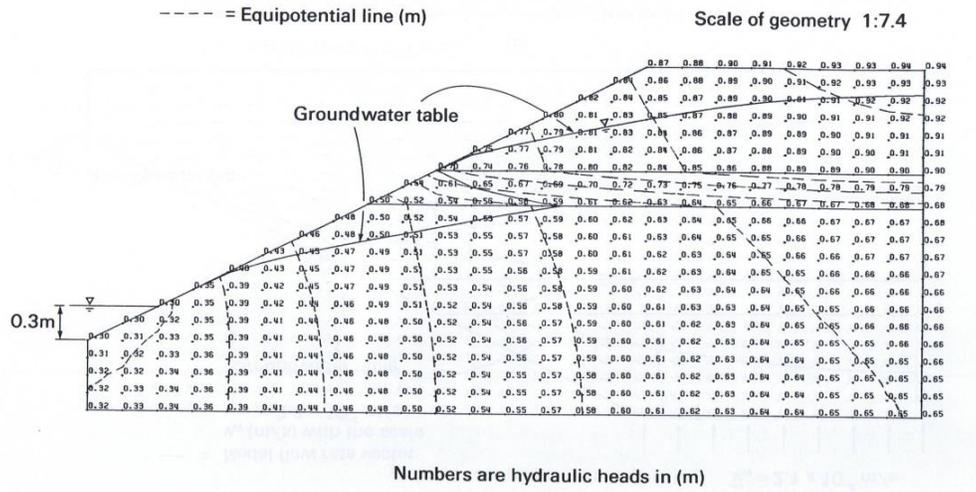
Figure 52 The geometry and boundary conditions of the model (1,370 nodes)

Two different kinds of materials, a medium sand and a fine sand with a lower hydraulic conductivity, are used for the model. The medium sand material is utilized for the bottom and top regions of the slope, and the fine sand material is used for the thin layer within the slope. Table 8 shows the detailed properties of the materials.

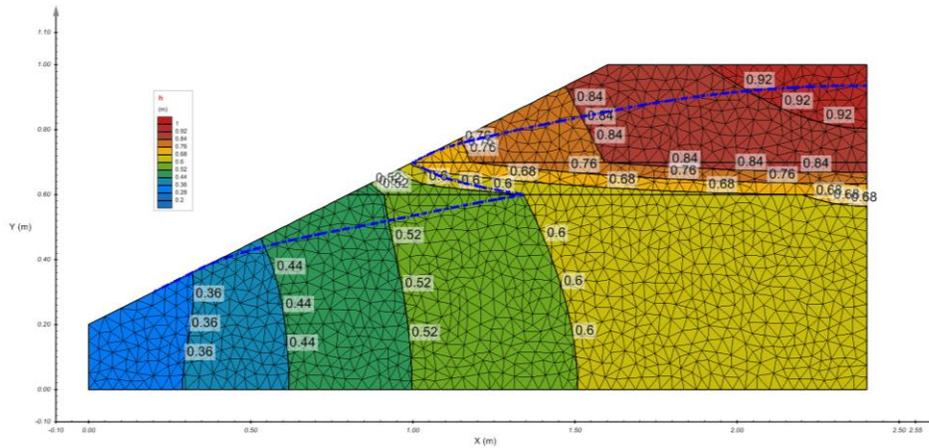
Table 8 Material properties used in the simulation

Tab	Parameters	Medium sand	Fine sand
New Material	Data Type	Unsaturated	Unsaturated
Volumetric Water Content	Saturated VWC	0.495	0.495
	SWCC	Fredlund and Xing Fit	Fredlund and Xing Fit
Hydraulic Conductivity	ksat (m/s)	1.4×10^{-3}	5.5×10^{-5}
	Unsaturated Hydraulic Conductivity	Fredlund, Xing and Huang Estimation	Fredlund, Xing and Huang Estimation

Figure 53 and Figure 54 show the total head (h) and pore water pressure (uw) results from Groundwater, and a physical comparison of the results of the Groundwater contours to those obtained in the Fredlund and Rahardjo (1993) textbook shows close agreement. The results are quantitatively compared with those from Fredlund and Rahardjo (1993) along the lines of $x = 1.6$ m and $y = 0.6$ m as shown in Figure 55 and Figure 56, and comparisons further verify the correctness of the results from Groundwater.



(a)



(b)

Figure 53 The comparison of total head contours between (a) Fredlund and Rahardjo (1993) and (b) Groundwater

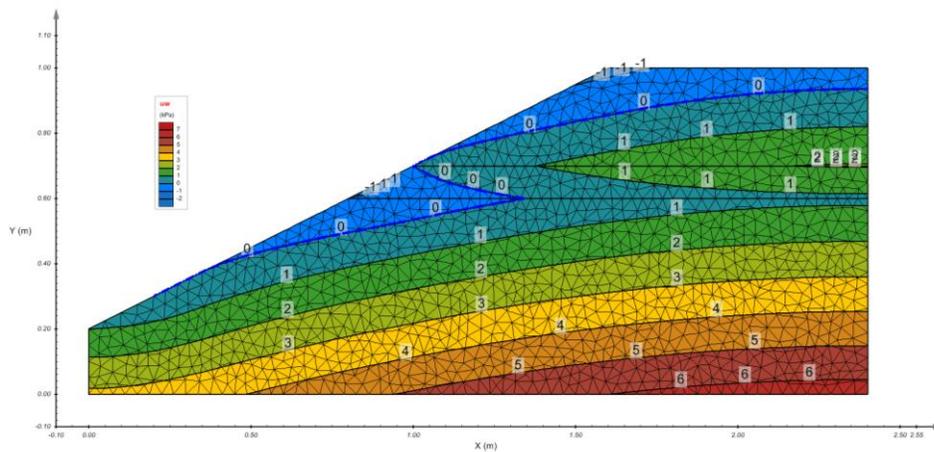


Figure 54 Pore water pressure contours from Groundwater

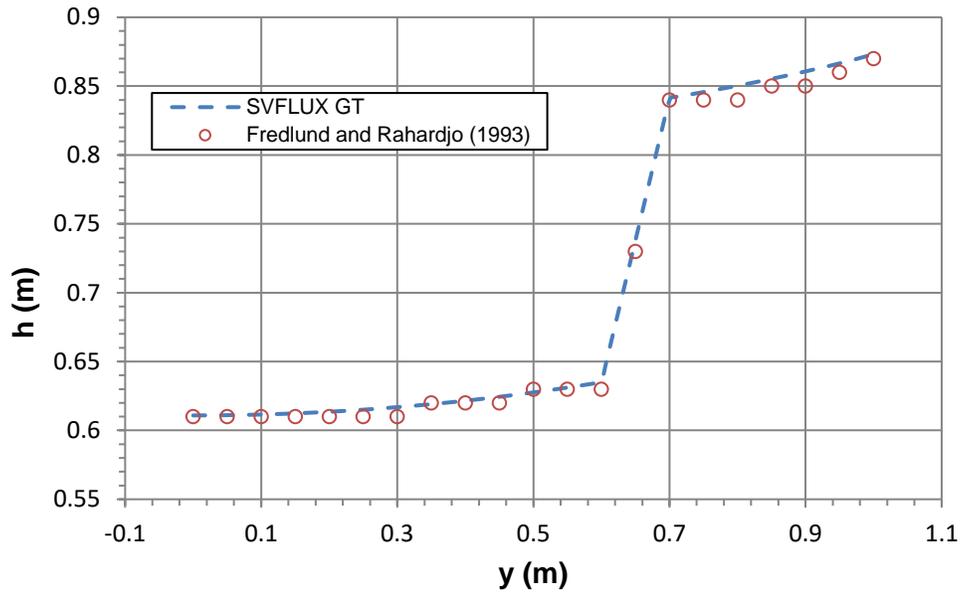


Figure 55 Comparison of total head results along the line of $x = 1.6$ m between Groundwater and Fredlund and Rahardjo (1993)

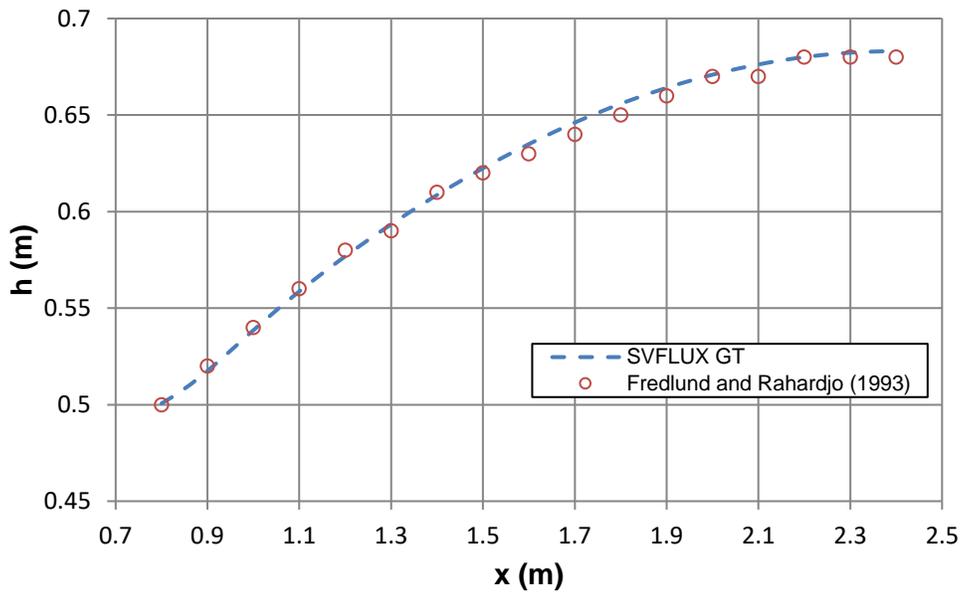


Figure 56 Comparison of total head results along the line of $y = 0.6$ m between Groundwater and Fredlund and Rahardjo (1993)

3.2 TRANSIENT STATE

A number of transient models were used to verify the Groundwater software. The following models demonstrate the successful ability of the Groundwater software to provide accurate transient solutions.

3.2.1 Transient Reservoir Filling

Project: EarthDams

Model: EarthDam_RF_steady_GT; EarthDam_RF_GT

The model involves the filling of a reservoir. This section begins with a brief description of the model followed by a comparison of the results between Groundwater and Pentland (2001).

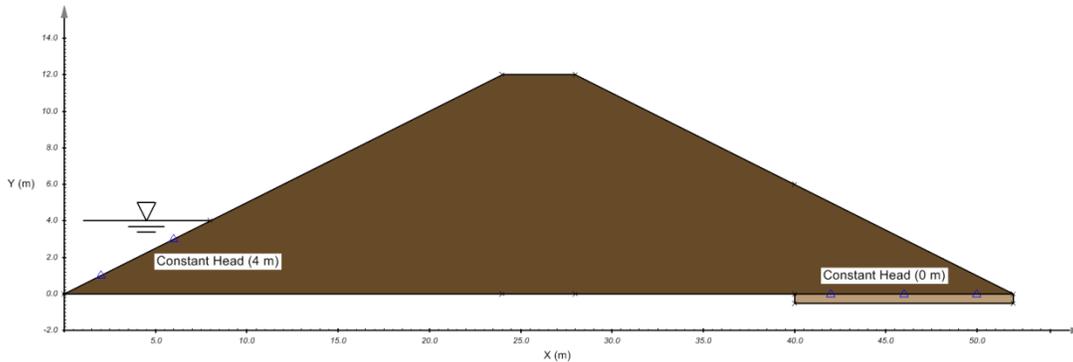


Figure 57 Geometry and boundary conditions for the steady-state solution in Groundwater

The earth fill dam considered is 12 m high 52 m in length and incorporates a filter on the downstream toe of the dam. The initial conditions of head were obtained by first solving a steady-state run of the model with the head on the upstream face of the dam set to 4 m and a head of 0 m on the lower portion of the filter as shown in Figure 57. All other boundaries were set to zero flow. The details of the material properties used for the dam are listed in Table 9. The results from the steady-state analysis were then imported as the initial conditions for the transient analysis as shown in Figure 58 and Figure 59.

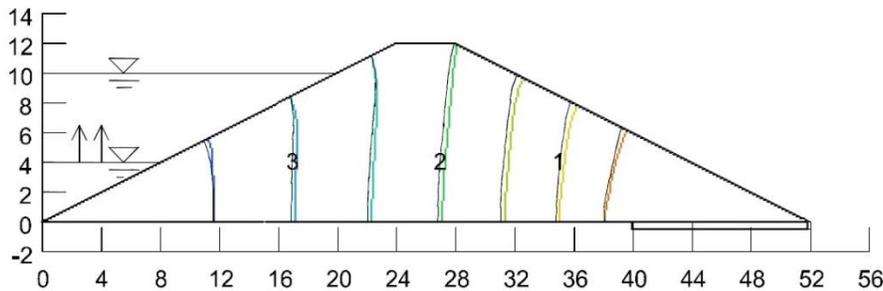


Figure 58 Head contours at the initial condition from Pentland (2001)

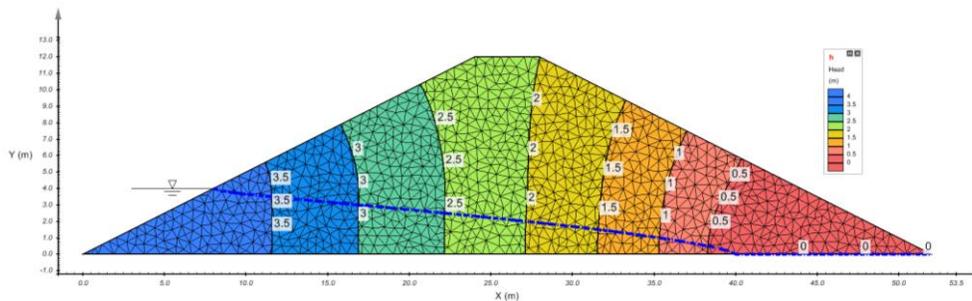


Figure 59 Head contours at the initial condition from Groundwater

Table 9 Details of material properties

Tabs	Parameters	Dam silt
New Material	Data Type	Unsaturated

	Saturated VWC	0.368
Volumetric Water Content	SWCC	Fredlund and Xing Fit
	Compressibility, m_v (1/kPa)	0.001
	k_{sat} (m/s)	10^{-7}
Hydraulic Conductivity	Unsaturated Hydraulic Conductivity	Modified Campbell Estimation
	Modified Campbell p	2.8

While the material properties remain the same in the transient flow model, the boundary conditions change slightly. A head of 10 m is set on the upstream face of the dam to simulate a full reservoir condition as shown in Figure 60. The model is run for 16,383 hours. Below, Figure 61, Figure 62, Figure 70 and Figure 71 show the head contours computed by SVFLUX GE and SEEP/W from Pentland (2001) and Groundwater at times of 15 and 16,383 hours. More results of head and pore-water pressure contours at times of 15, 225, 1,023, 4,095 and 16,383 hours from Groundwater are also provided.

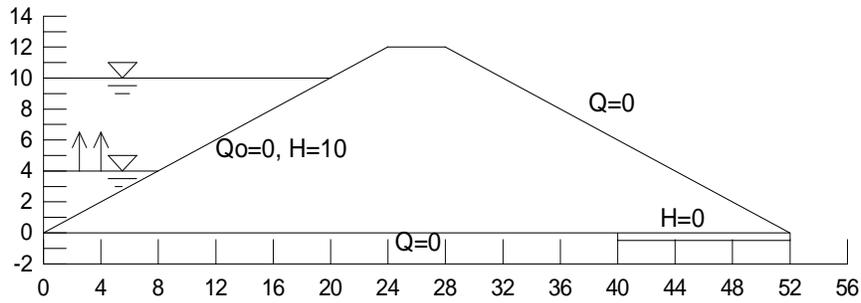


Figure 60 Reservoir filling description (Pentland, 2001)

• Results

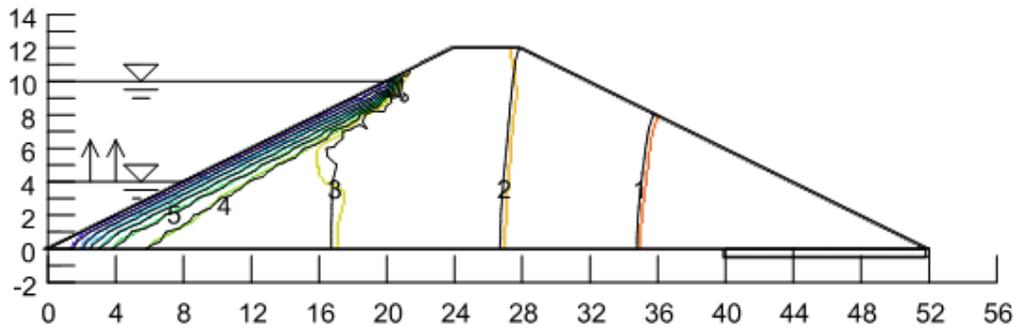


Figure 61 Computed head contours at time 15 hours from Pentland (2001) (Seep/W results in black, SVFLUX GE in color)

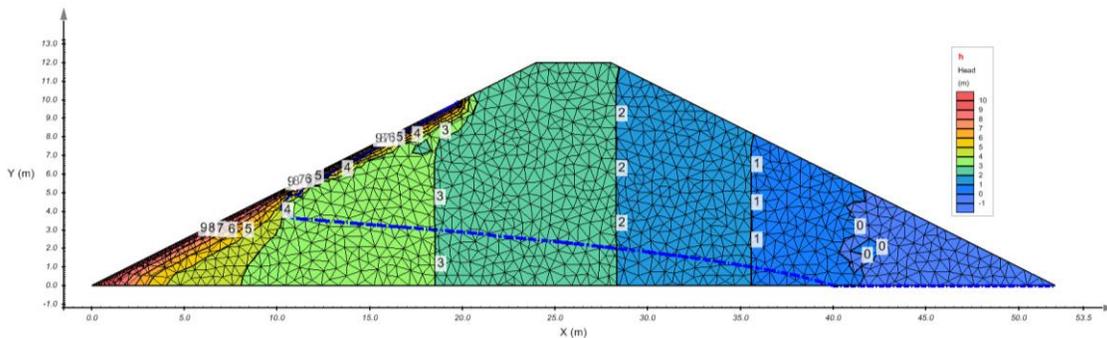


Figure 62 Computed head contours at time 15 hours from Groundwater

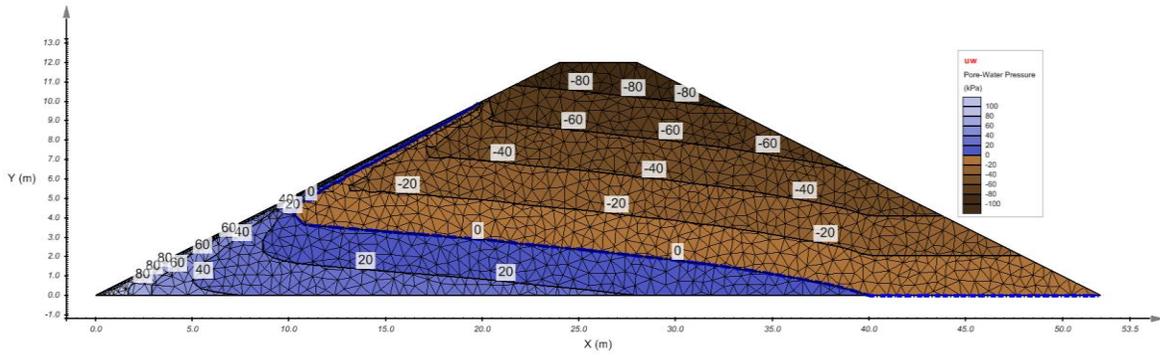


Figure 63 Computed pore water pressure contours at time 15 hours from Groundwater

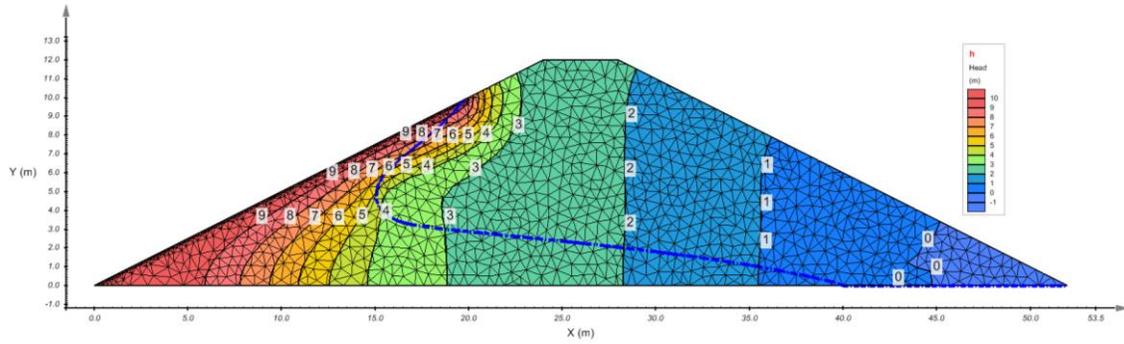


Figure 64 Computed head contours at time 225 hours from Groundwater

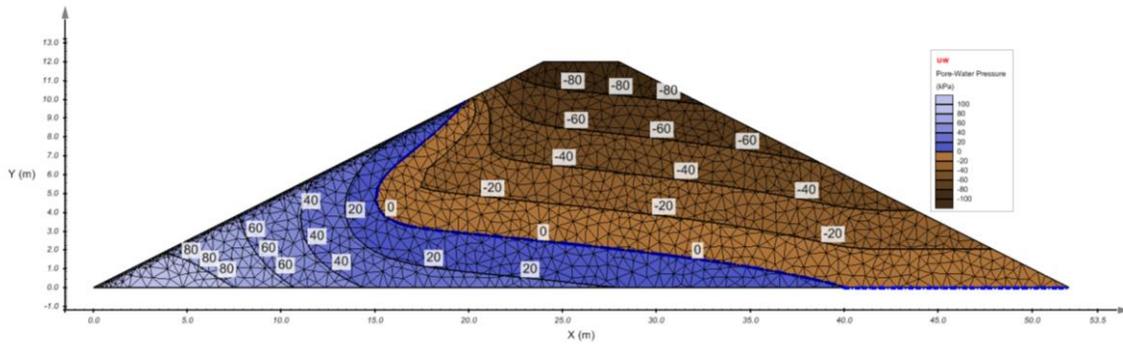


Figure 65 Computed pore water pressure contours at time 225 hours from Groundwater

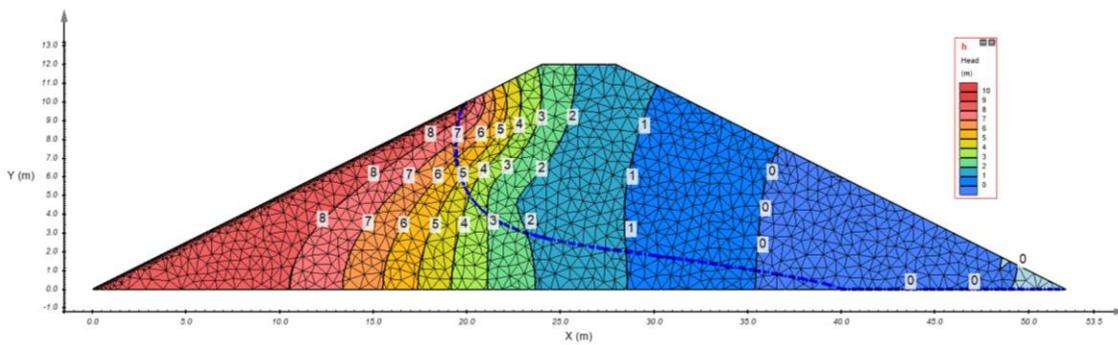


Figure 66 Computed head contours at time 1,023 hours from Groundwater

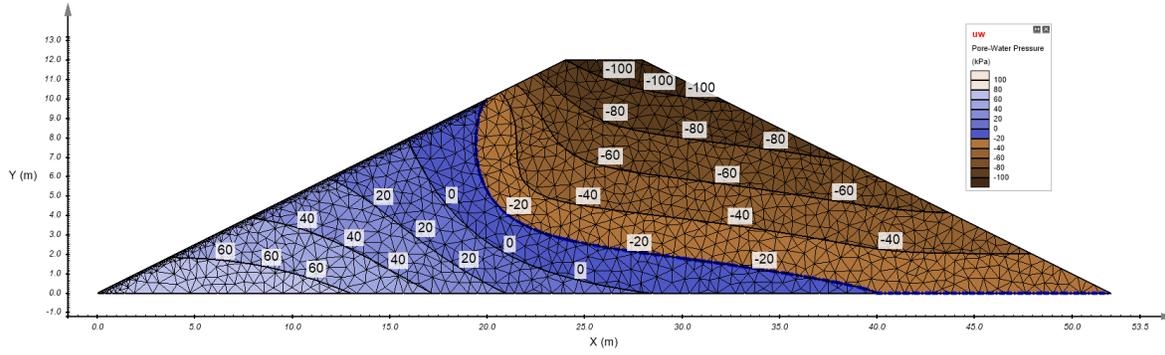


Figure 67 Computed pore water pressure contours at time 1,023 hours from Groundwater

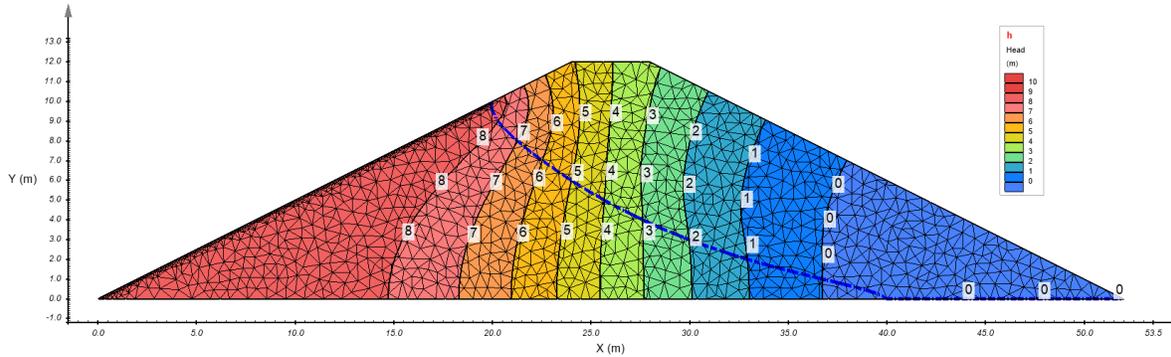


Figure 68 Computed head contours at time 4,095 hours from Groundwater

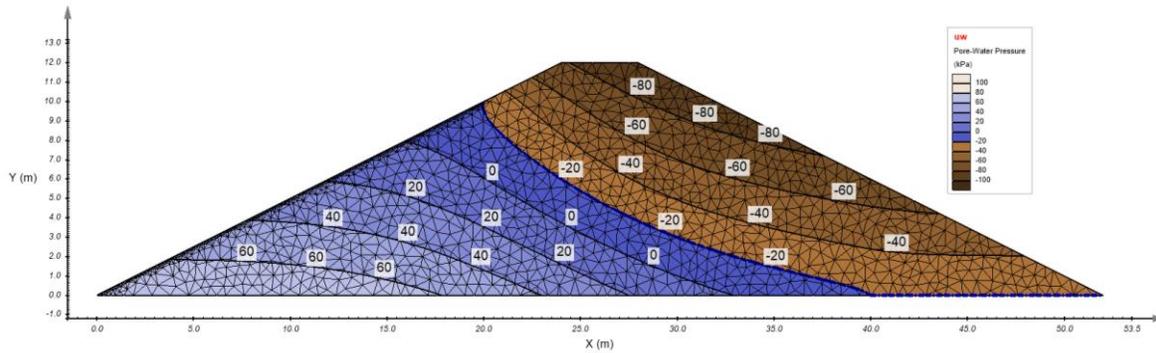


Figure 69 Computed pore water pressure contours at time 4,095 hours from Groundwater

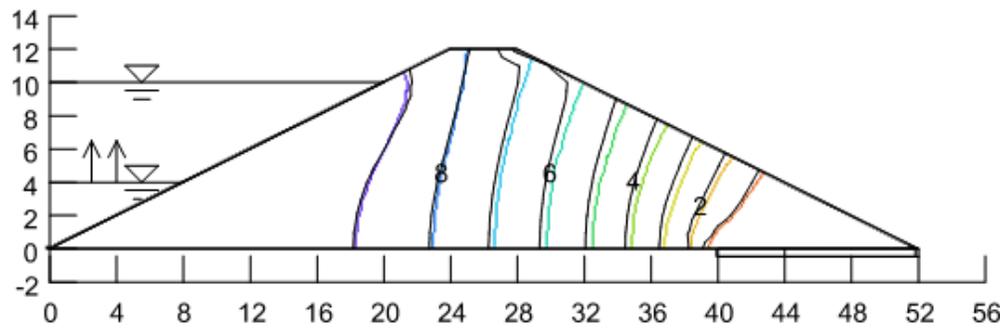


Figure 70 Computed head contours at time 16,383 hours from Pentland (2001) (Seep/W results in black, SVFLUX GE in color)

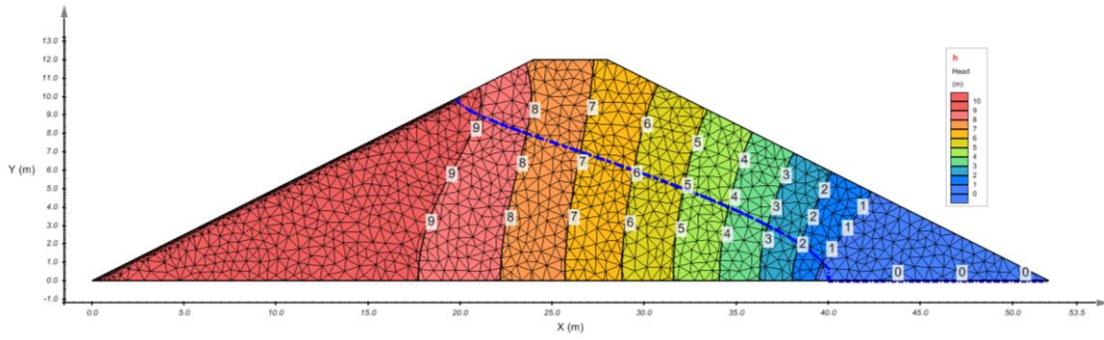


Figure 71 Computed head contours at time 16,383 hours from Groundwater

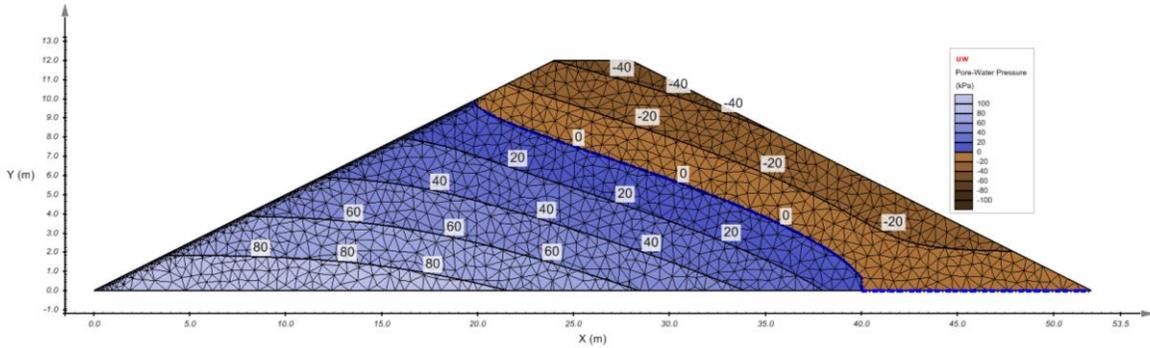


Figure 72 Computed pore water pressure contours at time 16,383 hours from Groundwater

It can be seen from the above figures that the results computed by Groundwater are in good agreement with those from Pentland (2001). Small differences appear and are likely due to differences in temporal and spatial discretization between different programs. This model further verifies the ability of Groundwater for simulating the seepage flow in transient problems.

3.2.2 Groundwater Seepage Below a Lagoon

Project: USMEP_Textbook
 Model: LagoonWithLiner_GT

This verification model illustrates unsteady-state groundwater seepage below a lagoon. The lagoon is placed on top of a 1 m thick soil liner, and the total height of the model is 10 m as shown in Figure 73. The geometry of the problem is symmetrical, and the liner and the surrounding soil are assumed to be isotropic with respect to their hydraulic conductivity.

An initial condition with a water table located 5 m below the ground surface is assumed. On the right boundary, a constant head (5 m) boundary condition is set below the water table, and the other surfaces are assumed as "Zero Flux". The lagoon is set as a constant pressure head (1 m) boundary condition to assume that the lagoon is filled with water to 1 m height at the time being equal to 0. More detailed descriptions about this example can be found in the book by Fredlund and Rahardjo (1993).

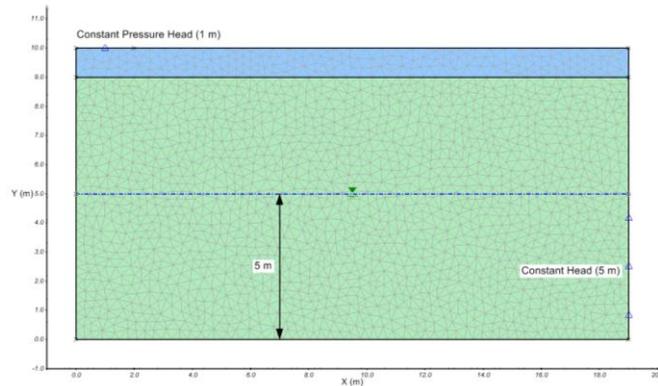


Figure 73 Geometry and boundary conditions of the groundwater seepage below a lagoon model (1,812 nodes)

Table 10 shows the details of material properties used in the example. The hydraulic conductivity function for the soil and soil liner are shown in Figure 74.

Table 10 Material properties used in the model

Tab	Parameters	Soil	Soil liner
New Material	Data Type	Unsaturated	Unsaturated
Volumetric Water Content	Saturated VWC	0.495	0.495
	Compressibility, m_v (1/kPa)	2×10^{-3}	2×10^{-3}
	ksat (m/hr)	3.6×10^{-2}	1.8×10^{-2}
Hydraulic Conductivity	Unsaturated Hydraulic Conductivity	Data	Data

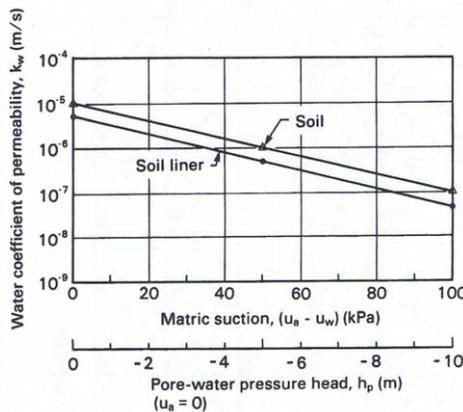
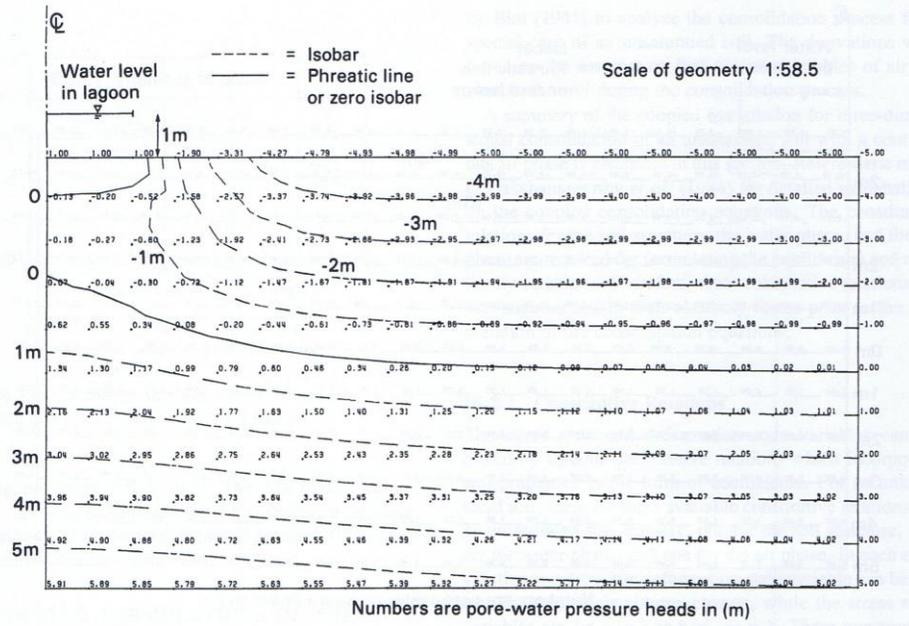
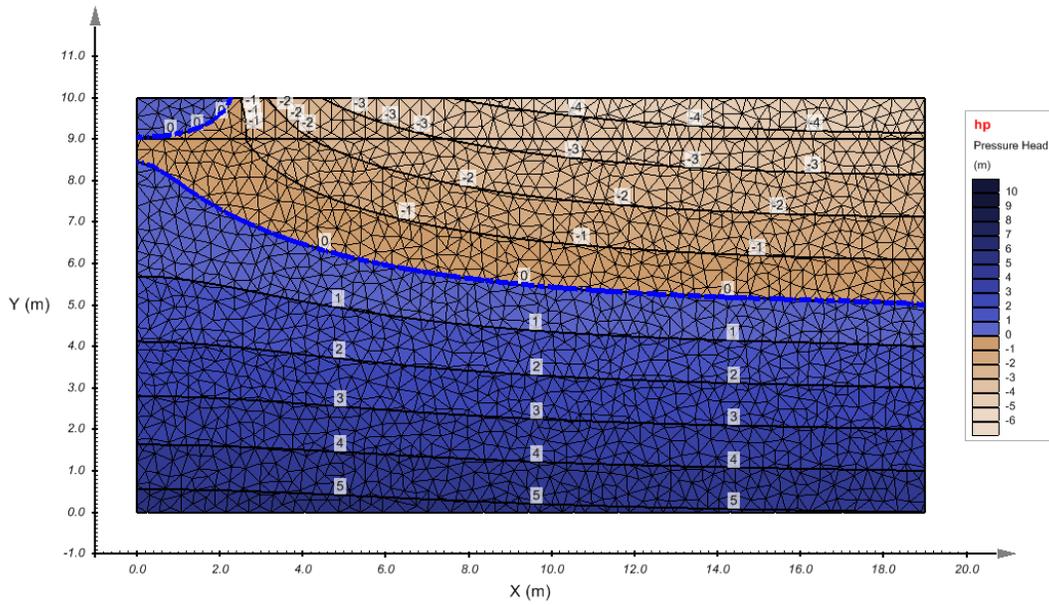


Figure 74 Hydraulic conductivity functions for the materials from Fredlund and Rahardjo (1993).

In this transient model, the solution is run for 200 hours, and it can be seen to reach the steady state at 189 hours according to Fredlund and Rahardjo (1993). Figure 75, Figure 76 and Figure 77 show the comparisons of pressure head contours from Fredlund and Rahardjo (1993) and Groundwater at the times of 7 hours, 13 hours and steady state. The results from Groundwater are in good agreement with those from Fredlund and Rahardjo (1993). This example further verifies the capability of Groundwater for seepage simulations in transient state.

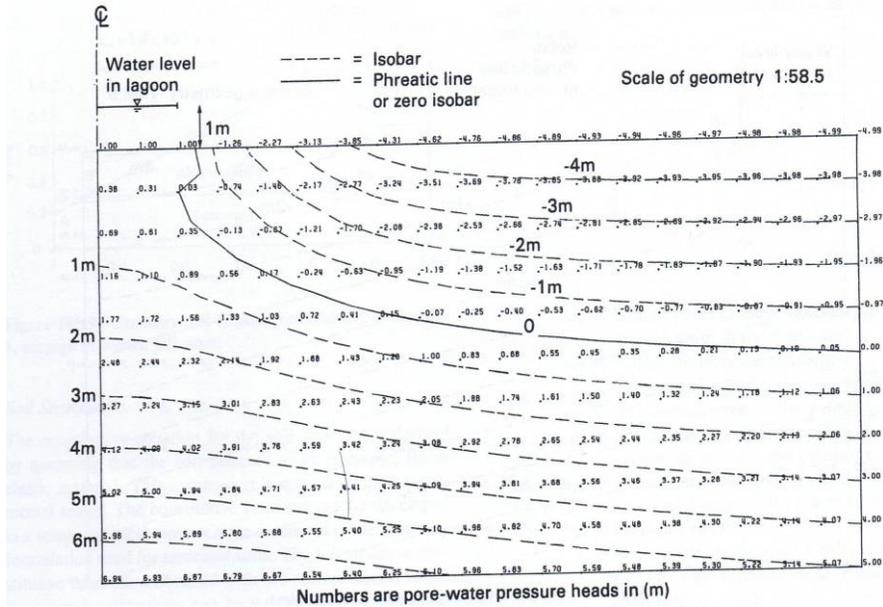


(a)

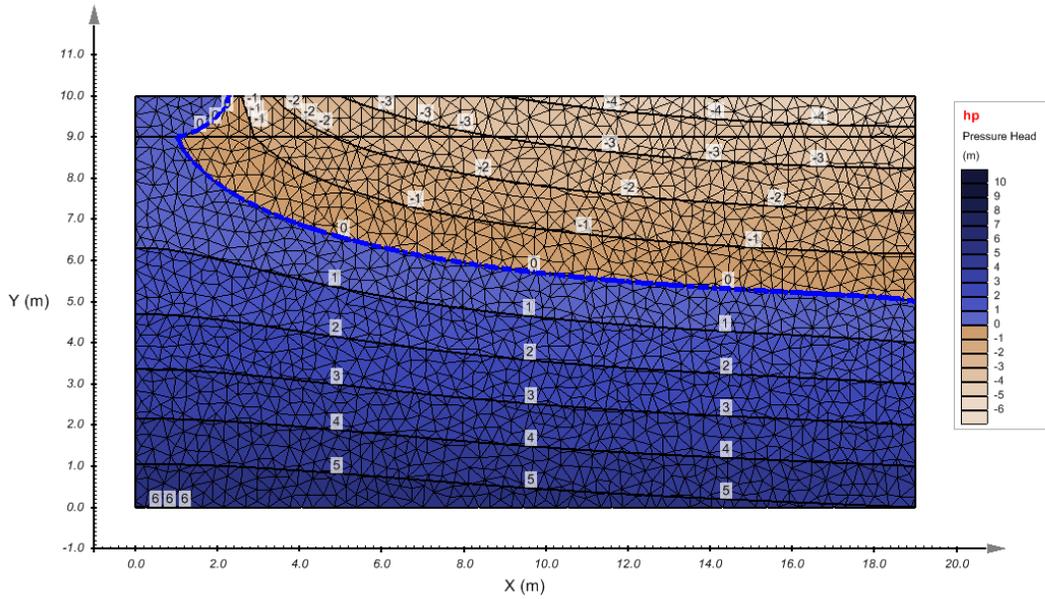


(b)

Figure 75 The comparison of pressure head contours from (a) Fredlund and Rahardjo (1993) and (b) Groundwater at 7 hours.

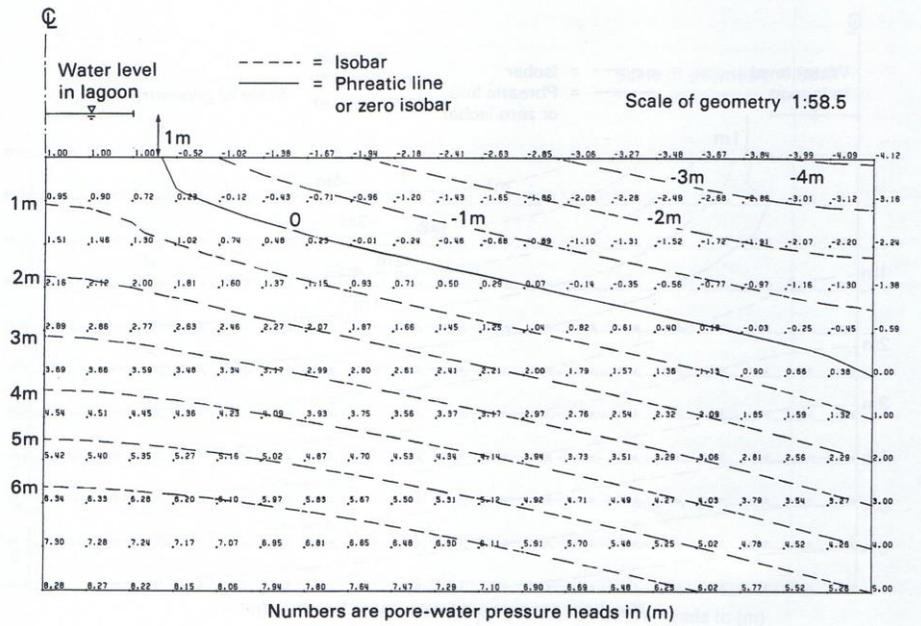


(a)

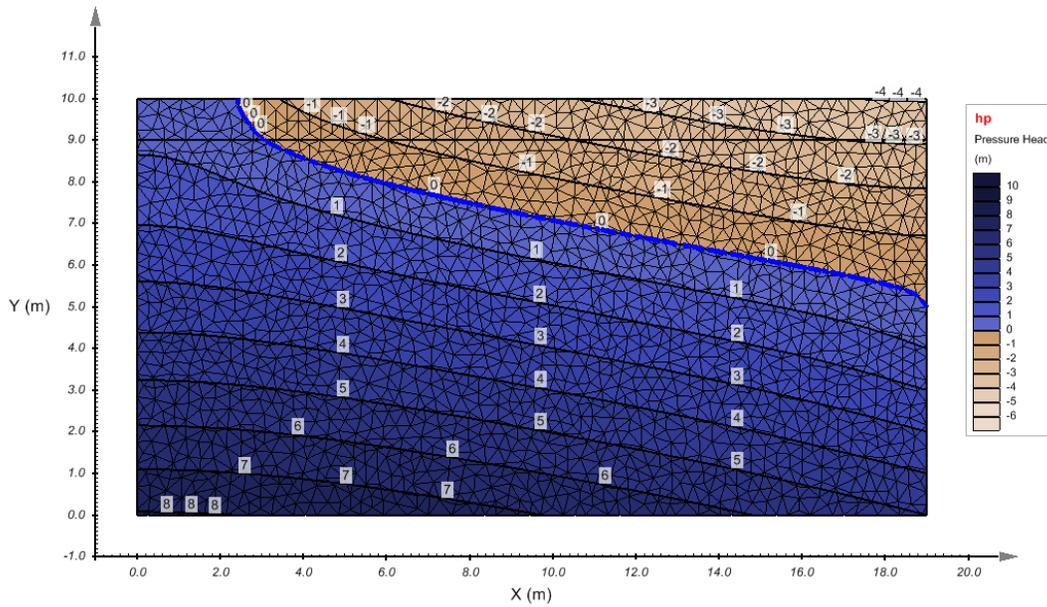


(b)

Figure 76 The comparison of pressure head contours from (a) Fredlund and Rahardjo (1993) and (b) Groundwater at 13 hours.



(a)



(b)

Figure 77 The comparison of pressure head contours from (a) Fredlund and Rahardjo (1993) and (b) Groundwater at the steady state.

3.2.3 Transient Phreatic Flow Subjected to Horizontal Seepage

Project: WaterFlow
 Model: HorizontalPhreaticFlow_GT

This verification model considers transient seepage through a fully confined aquifer. The aquifer is 100 m long and 5 m thick. The aquifer has 10 m Constant Head on the left side and 5 m Constant Head on the right side, and the bottom and top sides are viewed as Zero Flux. The initial condition is viewed as a Constant Head of 5 m in the aquifer. Seepage is then examined in the x-direction with time. The geometry and boundary conditions of the model are shown in Figure 78. A mesh of 2,150 nodes is created for this model.

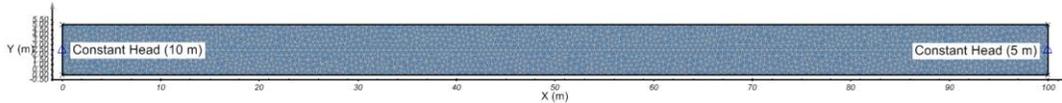


Figure 78 Geometry and boundary conditions of the model

The material is assumed to be saturated with a hydraulic conductivity of 1 m/hr and the compressibility of the system due to a change in pore-water pressure (m_v) of 0.1 1/kPa. Table 11 indicates the details of the material properties.

Table 11 Material properties used in the simulation

Tags	Parameters	Soil
New Material	Data Type	Saturated
Volumetric Water Content	Saturated VWC	0.4
Hydraulic Conductivity	ksat (m/hr)	1.0
Compressibility of the system	mv (1/kPa)	0.1

According to Tao and Xi (2006), in a semi-infinite aquifer bounded by a linear channel, during the period of that vertical seepage is equal to zero (or can be neglected) and that the channel-water stage is raised rapidly to an altitude, the transient phreatic flow-process is only affected by the horizontal seepage from the channel. The J. G. Ferris Formula can be obtained as:

$$h(x,t) = h(x,0) + \Delta H \cdot \operatorname{erfc} \left(\frac{x}{2\sqrt{\frac{k}{r_w \cdot m_v} t}} \right) \quad [3]$$

where

- $h(x,t)$ – the total head at position x at time t,
- $h(x,0)$ – the total head at position x at the initial condition (5 m),
- ΔH – the head difference between the initial head distribution and the introduced head (5 m),
- erfc – the complimentary error function.

Figure 79 shows the contour of total head at 600 hours. It can be seen that the values of head only change in the x-direction, so the vertical seepage can be neglected.

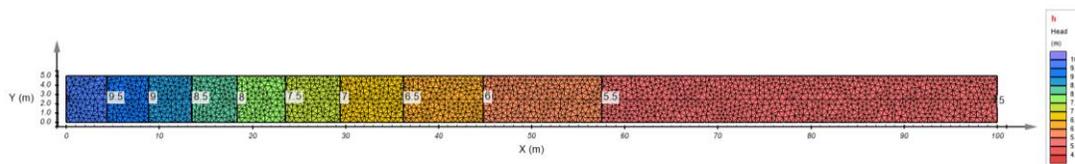


Figure 79 The contour of total head at 600 hours

A comparison of Groundwater results and the analytical calculations at different time (100 hours, 200 hours, 400 hours and 600 hours) is shown in Figure 80. The agreement between Groundwater and analytical results is good.

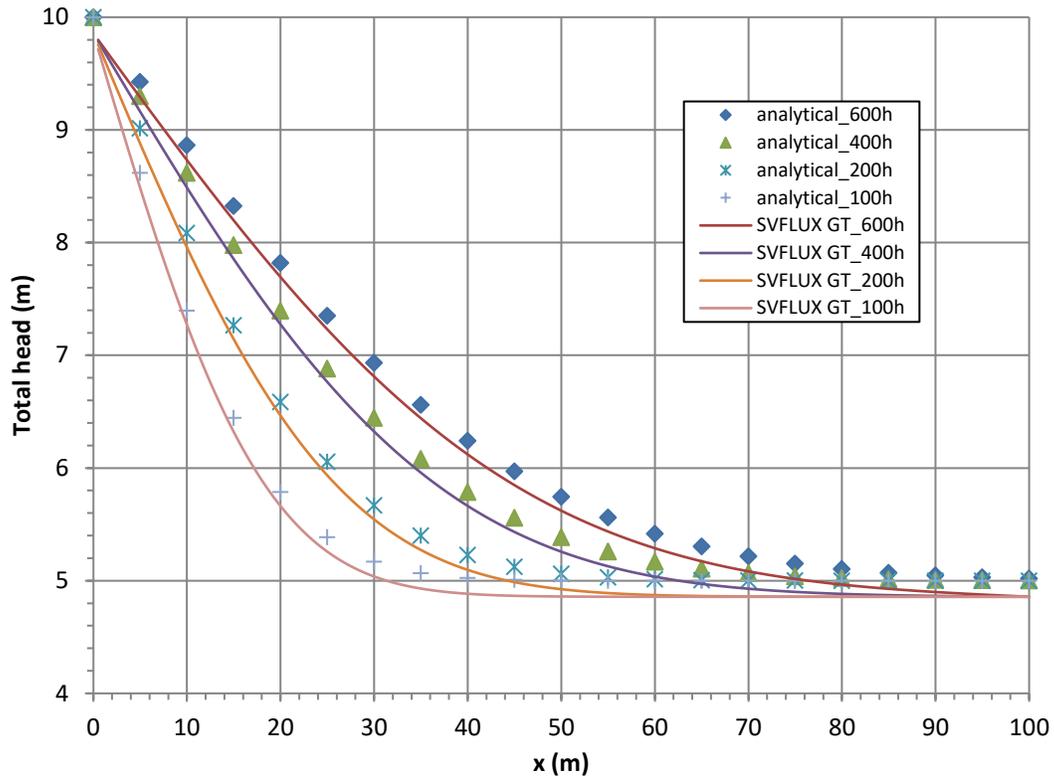


Figure 80 Comparison of Groundwater and analytical results

4 THREE-DIMENSIONAL SEEPAGE

Three-dimensional seepage models are presented in this chapter to provide a forum to compare the results of the Groundwater solver to the results of other seepage software and other documented examples. Both steady-state and transient models are considered.

4.1 STEADY-STATE

The following models are presented as steady-state verification when time is assumed to be infinite.

4.1.1 3D Reservoir

Project: Ponds

Model: Reservoir3D_DenseVerification_GT; Reservoir3D

This 3D steady-state example simulates the flow from a reservoir to a nearby river channel. The detailed geometry data, material properties and boundary conditions can be found in the SVFLUX GE tutorial manual. This model is re-created and solved using Groundwater with the same settings of the SVFLUX GE model, and the results from Groundwater are compared with those from SVFLUX GE. Figure 81 and Figure 82 shows the comparison of 3D contours of the total head between SVFLUX GE and Groundwater. The model results are in good agreement.

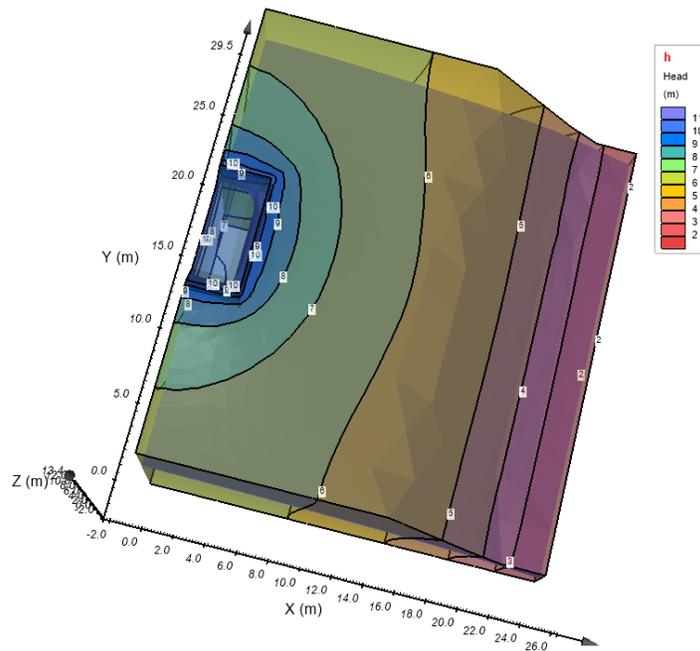


Figure 81 3D contours of total head from SVFLUX GE

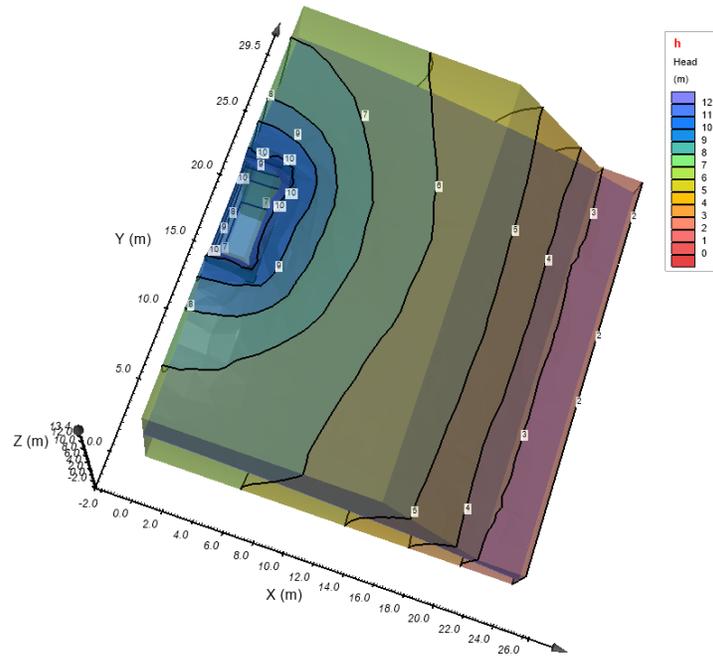


Figure 82 3D contours of total head from Groundwater

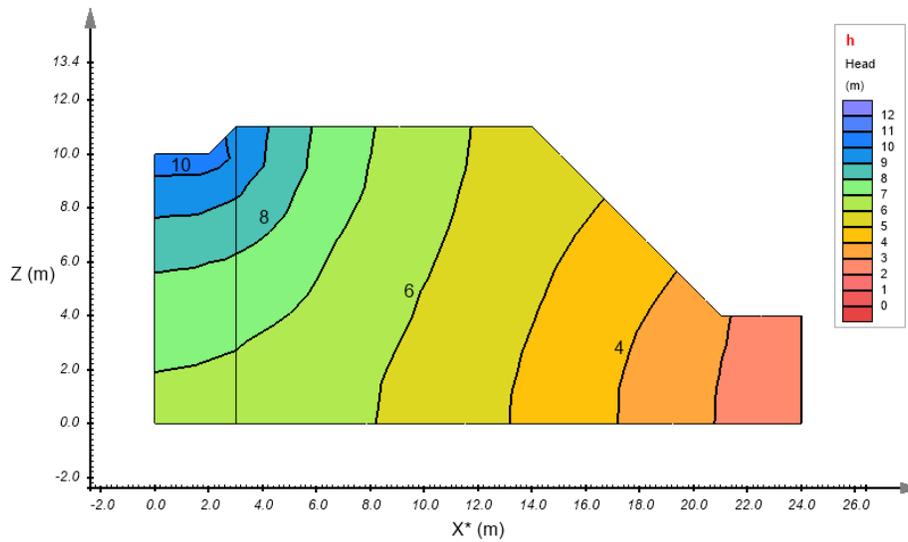


Figure 83 contours of total head at the slice surface (y = 13.5 m) from Groundwater

4.1.2 3D Pond

Project: Ponds
Model: Pstr01_DenseVerification_GT; Pstr01

The following example demonstrates a three dimensional seepage example that models the steady-state water flow from a pond to a body of water at the bottom of a slope. This is a tutorial model from SVFLUX GE tutorial manual.

The model geometry contains 3 regions and two surfaces. The regions consist of a pond, lake, and slope. The top surface is a grid of elevations that define the slope. The bottom surface has a constant elevation. A saturated soil is used to model the slope. The pond and lake are assigned constant head boundary conditions equal to their respective elevations. Figure 84 and Figure 85 show the contours of pore water pressure from SVFLUX GE and Groundwater, respectively, and they are in good agreement. This example verifies the ability of Groundwater in simulating fully three dimensional seepage problems.

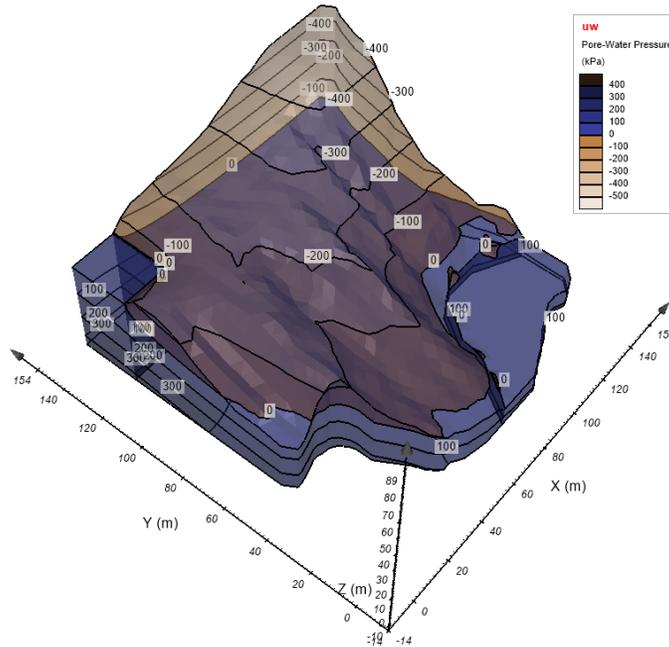


Figure 84 contours of pore water pressure from SVFLUX GE

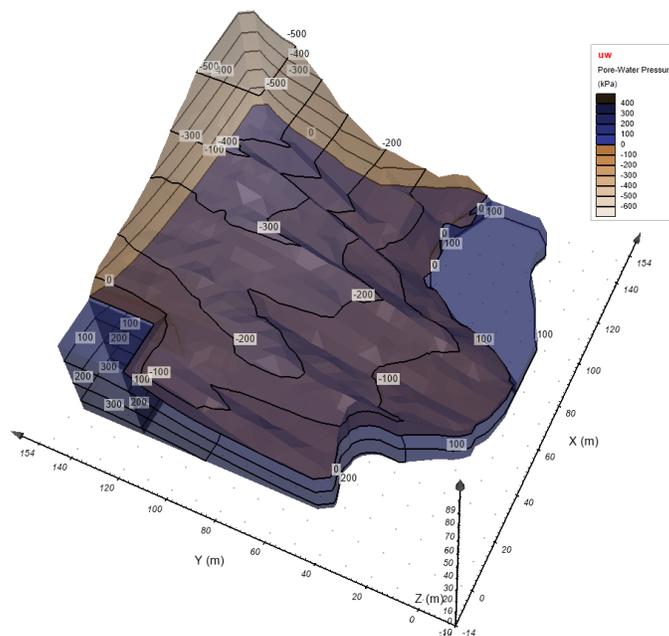


Figure 85 contours of pore water pressure from Groundwater

4.2 TRANSIENT STATE

A number of transient models were used to verify the Groundwater software. The following models demonstrate the successful ability of the Groundwater software to provide accurate transient solutions.

4.2.1 Transient Reservoir Filling 3D

Project: EarthDams

Model: EarthDat_RF_Steady_3D_GT; EarthDam_RF_3D_GT

This three-dimensional seepage verification model considers the transient seepage through an earth dam. The basic problem is similar to what has been considered in the 2D problem. The material properties are the same as those used in the 2D model as shown in Table 12. The 3D geometry of the earth fill dam is still 12 m high, 52 m in length and extruded from the 2D model with a width of 20 m. The initial conditions of head were obtained by first solving a steady-state run of the model with the head on the upstream face of the dam set to 4 m and a head of 0 m on the lower portion of the filter. The results from the steady-state analysis were then imported as the initial conditions for the transient analysis as shown in Figure 86 and Figure 87. This benchmark tests the ability of the software to solve a 3D model extruded from a 2D model and obtain the same results as that of the 2D model.

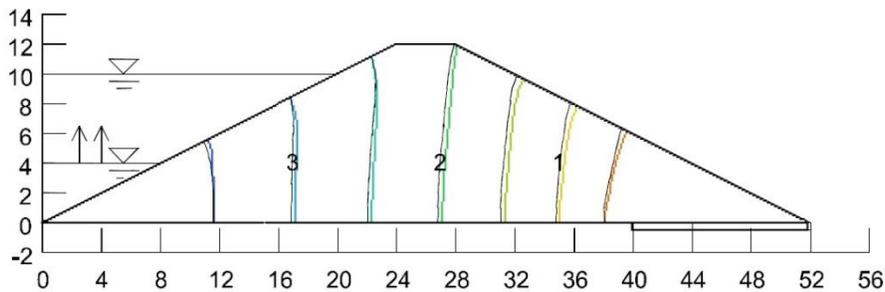


Figure 86 Head contours at the initial condition from Pentland (2001)

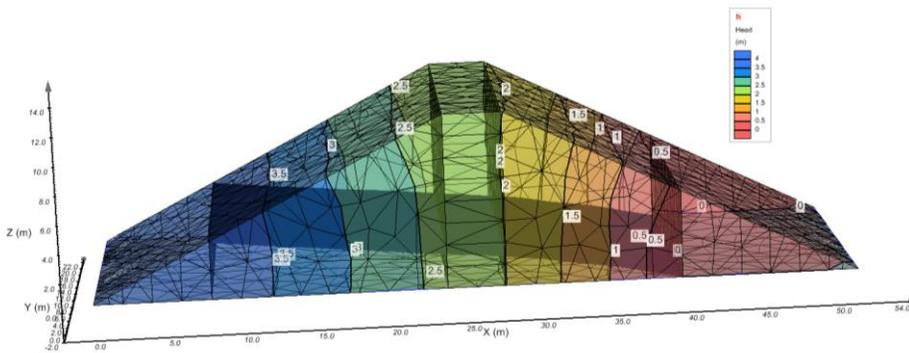


Figure 87 Head contours at the initial condition from Groundwater

Table 12 Details of material properties

Tabs	Parameters	Dam silt
New Material	Data Type	Unsaturated
	Saturated VWC	0.368
Volumetric Water Content	SWCC	Fredlund and Xing Fit
	Compressibility, m_v (1/kPa)	0.001
	ksat (m/s)	1e-7
Hydraulic Conductivity	Unsaturated Hydraulic Conductivity	Modified Campbell Estimation
	Modified Campbell p	5

While the material properties remain the same in the transient flow model, the boundary conditions change slightly. A head of 10 m is set on the upstream face of the dam to simulate a full reservoir condition as shown in Figure 88. The model is run for

16,383 hours. Below, Figure 89, Figure 90, Figure 98 and Figure 99 show the head contours from Pentland (2001) and Groundwater at times of 15 and 16,383 hours, and more results of 3D head and pore-water pressure contours at times of 15, 225, 1,023, 4,095 and 16,383 hours from Groundwater are also provided.

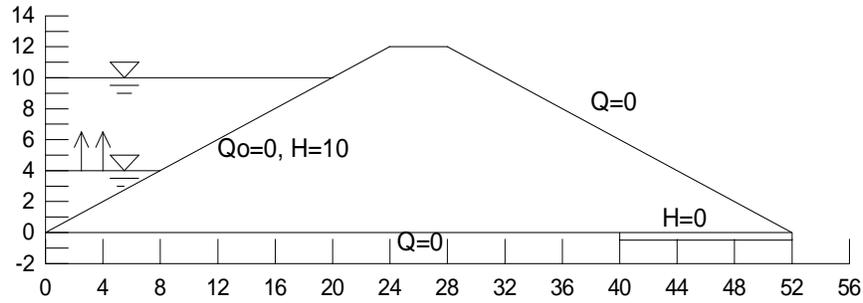


Figure 88 Reservoir filling description (Pentland, 2001)

• Results

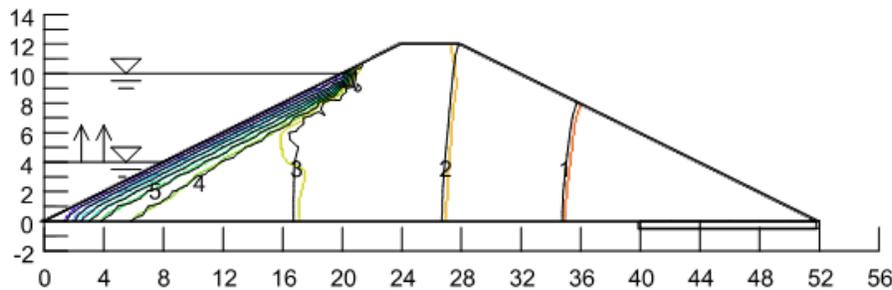


Figure 89 Computed head contours at time 15 hours from Pentland (2001) (Seep/W results in black, SVFLUX GE in color)

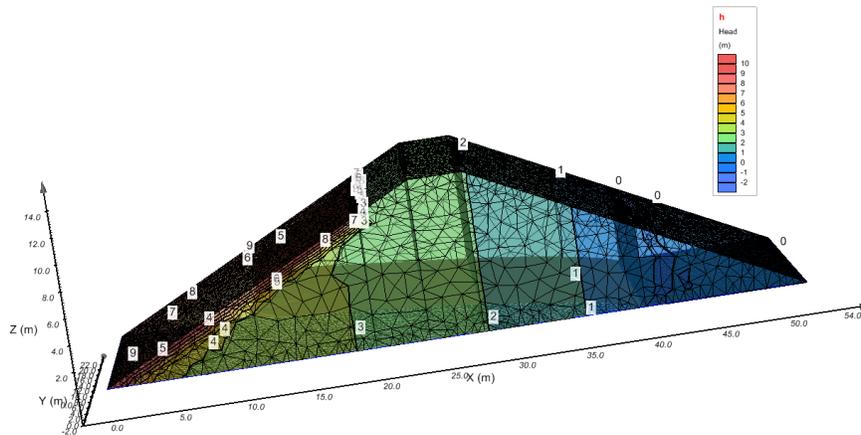


Figure 90 3D head contours at time 15 hours from Groundwater

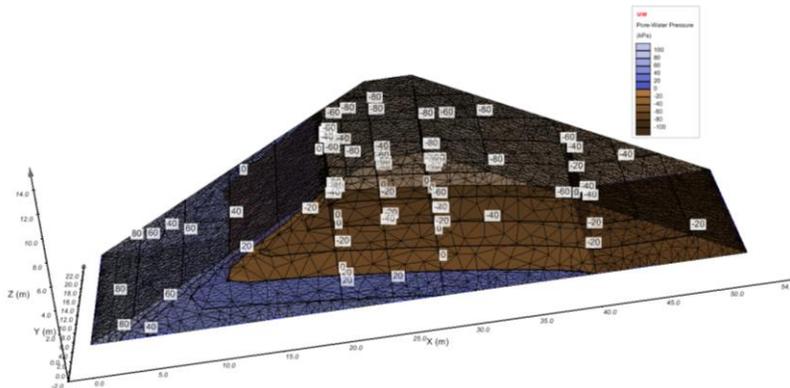


Figure 91 3D pore-water pressure contours at time 15 hours from Groundwater

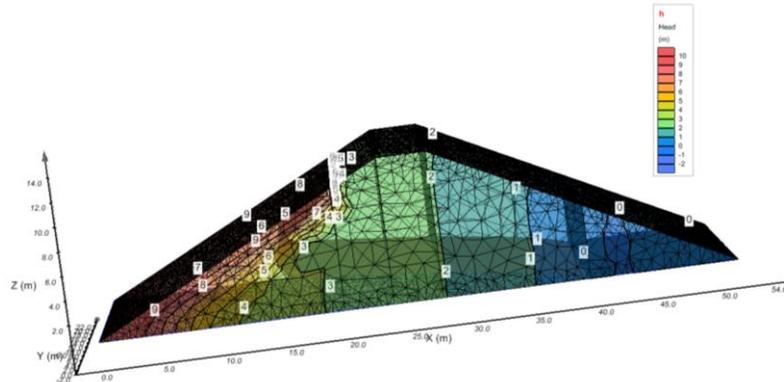


Figure 92 3D head contours at time 225 hours from Groundwater

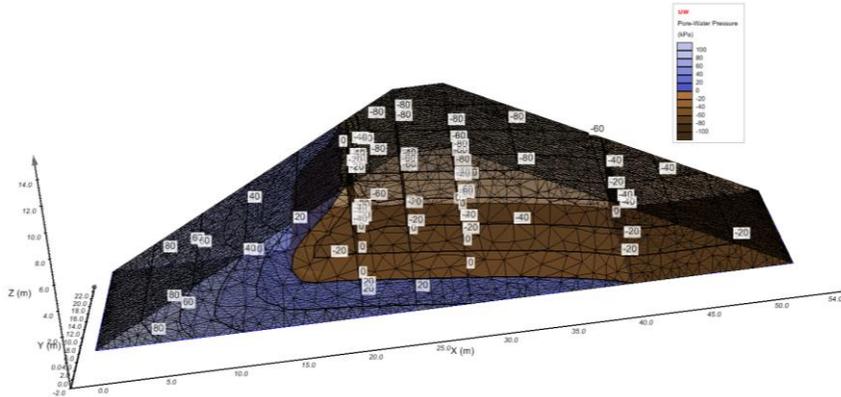


Figure 93 3D pore-water pressure contours at time 225 hours from Groundwater

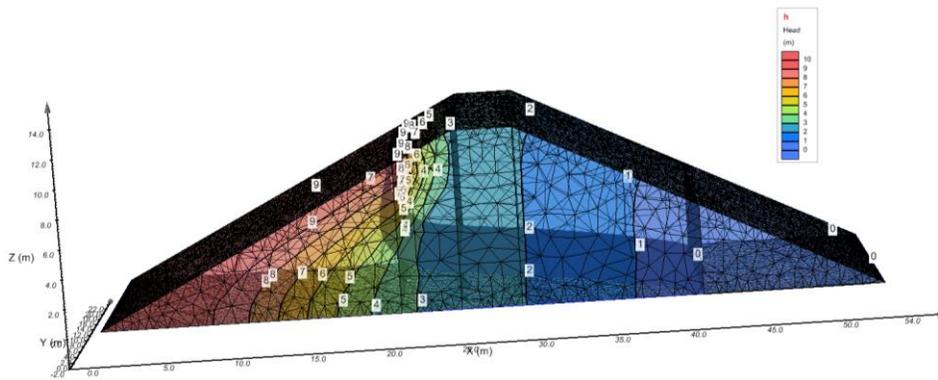


Figure 94 3D head contours at time 1,023 hours from Groundwater

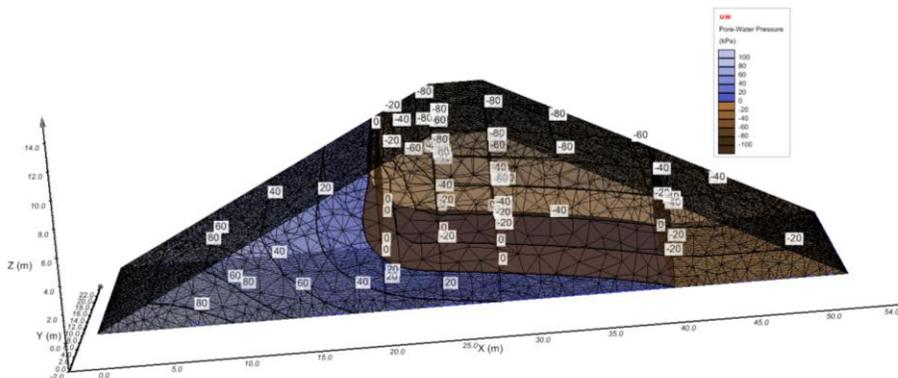


Figure 95 3D pore-water pressure contours at time 1,023 hours Groundwater

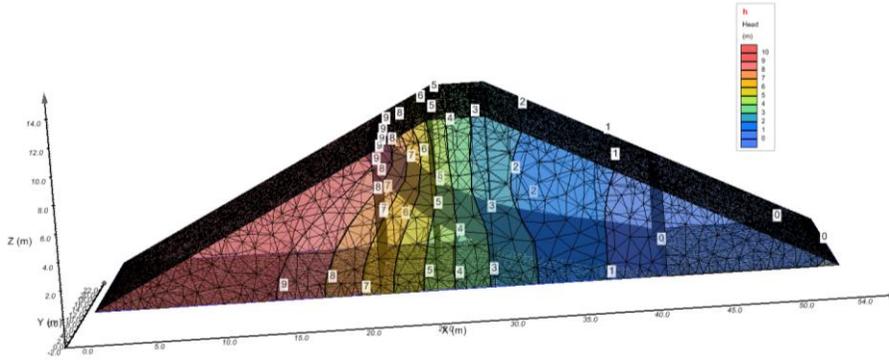


Figure 96 3D head contours at time 4,095 hours Groundwater

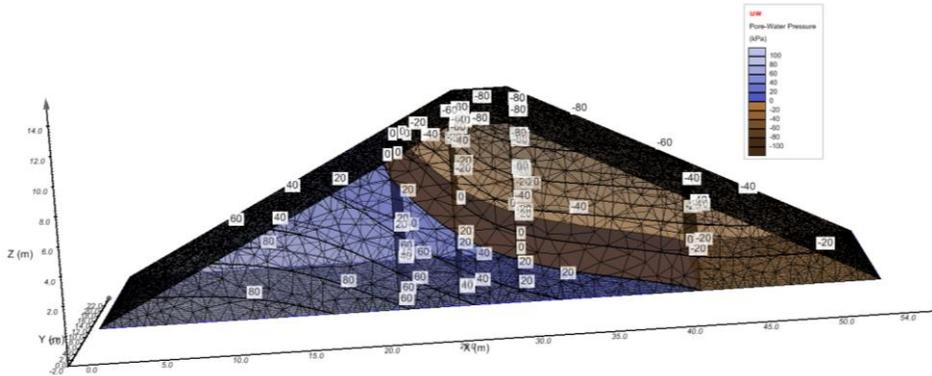


Figure 97 3D pressure contours at time 4,095 hours from Groundwater

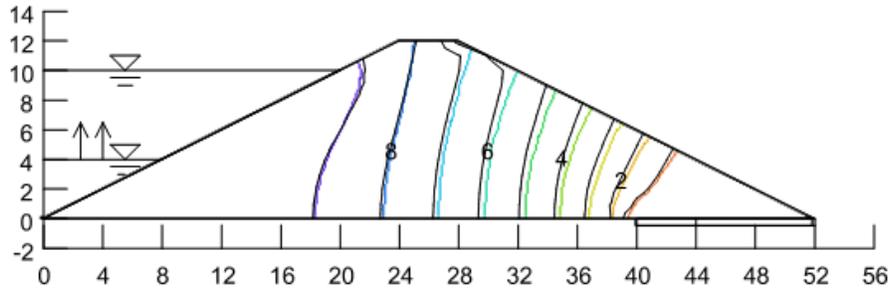


Figure 98 Computed head contours at time 16,383 hours from Pentland (2001) (Seep/W results in black, SVFLUX GE in color)

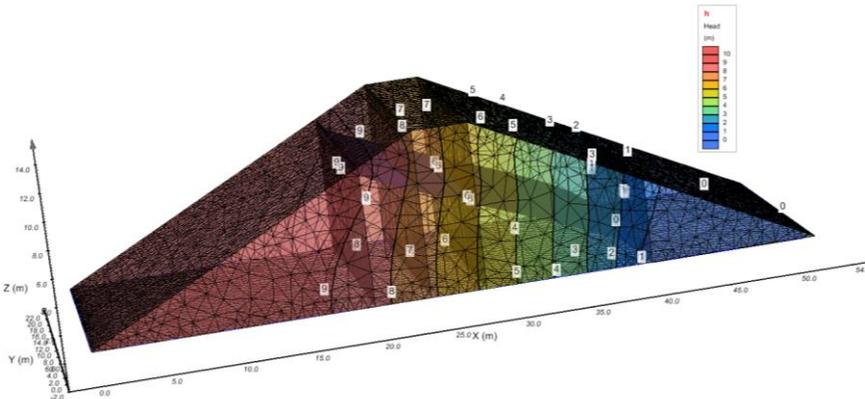


Figure 99 3D head contours at time 16,383 hours from Groundwater

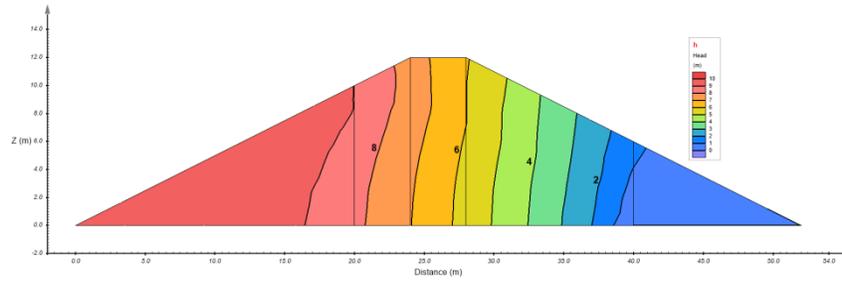


Figure 100 2D head contours at time 16,383 hours at the slice surface (y = 10 m) from Groundwater

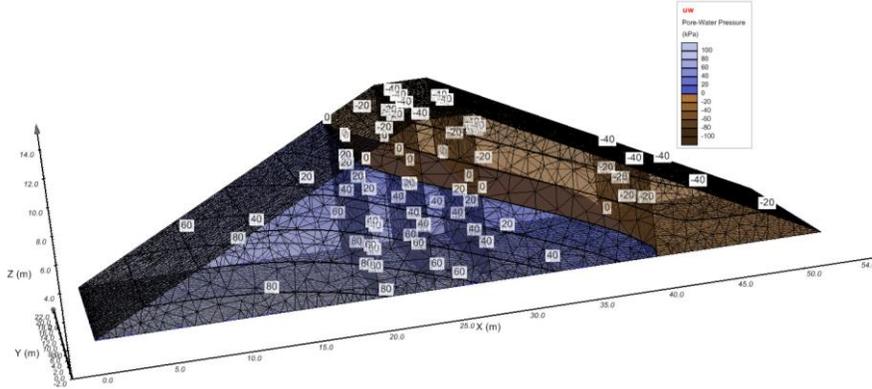


Figure 101 3D pore-water pressure contours at time 16383 hours from Groundwater

It can be seen from the above figures that the results computed by Groundwater are in good agreement with those from Pentland (2001). This model further verifies the ability of Groundwater for simulating the transient seepage flow in three-dimensional problems.

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