# NON-LINEAR DEFORMATION OF STRUCTURAL ELEMENTS

This document verifies that the non-linear deformation of plates is treated correctly in PLAXIS. An elastoplastic '$M - \kappa$' material type is used and the deformation of a cantilever under various loading conditions is studied.

Used version:

- PLAXIS 2D - Version 2018.0

**Geometry:** In PLAXIS structures cannot be used individually. A soil cluster is used to create the geometry. Note that the properties of the material assigned to the soil do not affect the results as the clusters will be deactivated in the calculation phases.

In PLAXIS 2D a plane strain model is used with 15-noded elements. An horizontal beam with length equal to 1 m is simulated. A *Point displacement* with fixed translations and rotation is used to clamp the beam at the left edge, while the right edge is free. A *Point moment* is used to apply bending moment to the free end. Figure 1 illustrates the model geometry.

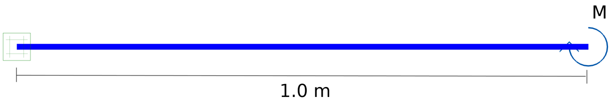

Figure 1 Model geometry in PLAXIS 2D

**Materials:** The *Elastoplastic* $M - \kappa$ material type is used, so the bending of the plate is defined in terms of a bending moment-curvature diagram. Table 1 presents the selected input values for the $M - \kappa$ diagram and Figure 2 depicts the diagram.

Table 1 Input values for the $M - \kappa$ diagram

| M [kNm] | $\kappa$ [1/m] |
|---|---|
| 0 | 0.0 |
| 25 | $1.786 \times 10^{-4}$ |
| 35 | $3.572 \times 10^{-4}$ |
| 39 | $5.358 \times 10^{-4}$ |
| 40 | $7.144 \times 10^{-4}$ |

The rest of the selected plate material properties are:

Plate: Isotropic $EA = 14 \times 10^6$ kN/m $w = 0$ kN/m/m $\nu = 0$

**Meshing:** The *Fine* option is selected for the *Element distribution*, with a global *Coarseness factor* equal to 1.

**Calculations:** In the Initial phase, zero initial stresses are generated by using the *K0 procedure* ($\gamma = 0$). In the following phases, the *Calculation type* is set to *Plastic analysis* and the soil cluster is deactivated, while the plate, the point fixity and the point moment are activated. A bending moment $M$, varying per calculation phase, is assigned to the point. The horizontal and vertical force $F$ is set to 0. In order to let the right end of the beam move freely, the *Deformations* option is deactivated in the *Model conditions*.

To enhance results accuracy in every calculation phase (excluding the Initial phase), the

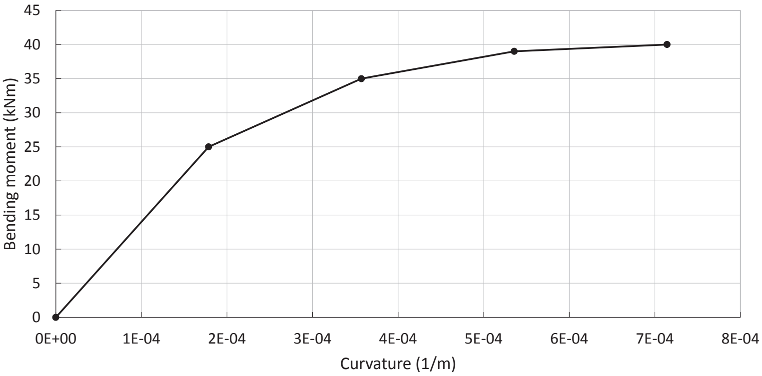

Figure 2 $M - \kappa$ diagram

*Max load fraction per step* equals 0.05.

In total 15 calculation phases are introduced after the Initial phase, which correspond to five 'load cases'. In the first two load cases, the plate is loaded to failure in both clockwise and anticlockwise directions. In the subsequent load cases various loading conditions are examined, in which the plate is loaded, unloaded and reloaded. Table 2 presents all the considered load cases in respect with the corresponding calculation phases. Note that some phases start from the Initial phase instead of the previous one and the options *Reset displacements to zero* and *Reset small strain* are activated. This is relevant for every phase at the beginning of a new load case, as presented in Table 2.

Table 2 Loading conditions

| Load case | Phase | M [kNm/m] | Comment |
|---|---|---|---|
| 1 | 1 | -41 | Loading to failure (clockwise) |
| 2 | 2 | +41 | Loading to failure (anticlockwise) |
| 3 | 3 | -25 | Loading |
|  | 4 | 0 | Unloading |
|  | 5 | -35 | Reloading |
|  | 6 | 0 | Unloading |
|  | 7 | -41 | Reloading to failure |
| 4 | 8 | -25 | Loading |
|  | 9 | +25 | Reverse loading |
|  | 10 | -35 | Reloading |
|  | 11 | +25 | Reverse reloading |
|  | 12 | -41 | Reloading to failure |
| 5 | 13 | -39.5 | Loading |
|  | 14 | +39.5 | Reverse loading |
|  | 15 | -39.5 | Reloading |

**Output:** The vertical displacement at the free end of the beam is plot against the bending moment developed at the support on the left edge. Figure 3 through 6 present the results of each load case. Note that, for the sake of comparison, load cases 1 and 2 are plotted together in Figure 3.

**Verification:** Analyzing the simple bending of a beam, the vertical deformation *u* at the

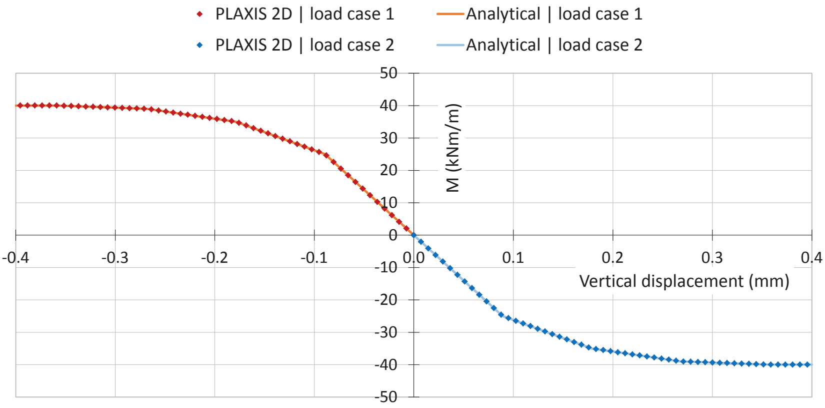

Figure 3  Bending moment versus vertical displacement (load cases 1 and 2)

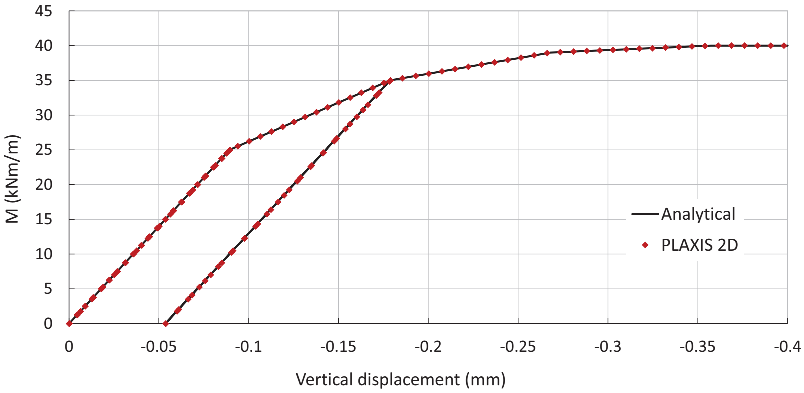

Figure 4  Bending moment versus vertical displacement (load case 3)

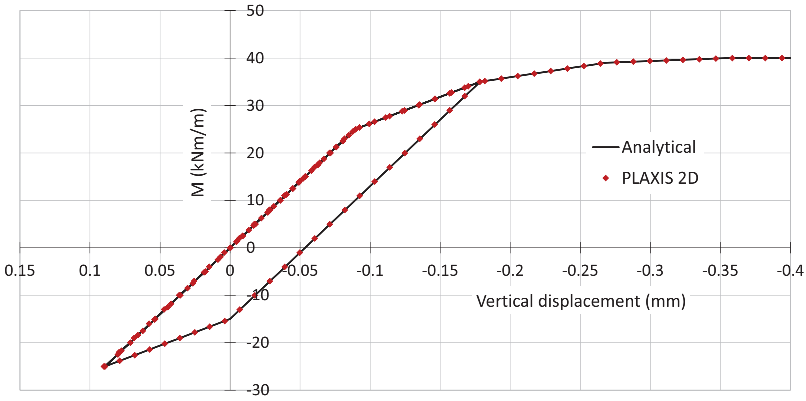

Figure 5  Bending moment versus vertical displacement (load case 4)

free end is calculated according to the small deformation theory:

$$M = EI\kappa = EI\frac{\partial^2 u}{\partial x^2} \tag{1}$$

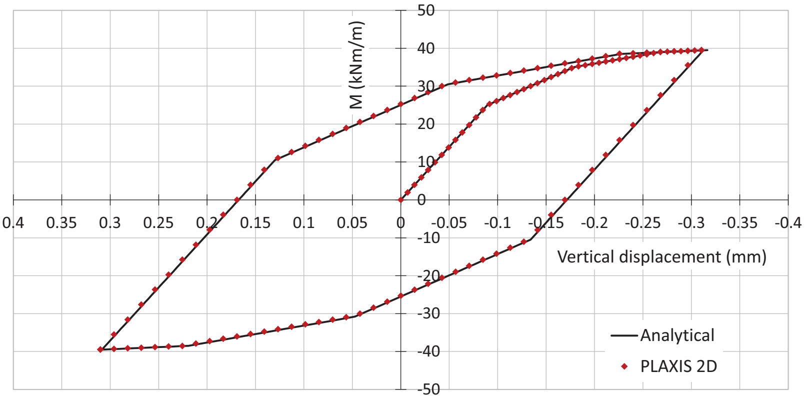

Figure 6 Bending moment versus vertical displacement (load case 5)

in which flexural rigidity $EI$ varies with the applied bending moment $M$, according to the $M - \kappa$ diagram.

Based on Eq. (1) and Table 1, curvature $\kappa$ is calculated as:

$$\kappa_i = \kappa_{i-1}^{max} + \frac{M_i - M_{i-1}^{max}}{EI_i} \tag{2}$$

where the notation $i$ refers to the corresponding branch of the $M - \kappa$ diagram, thus $i$ = [1..4]. The elastic branch corresponds to $i$ = 1. Curvature $\kappa_{i-1}^{max}$ and bending moment $M_{i-1}^{max}$ refer to the maximum values reached at the previous branch ($i - 1$) during loading or unloading.

The corresponding vertical displacement $u$ is obtained as:

$$u = \int\int \kappa \, dxdx = \kappa \frac{L^2}{2} \tag{3}$$

Figure 3 presents the results of the analytical solution against PLAXIS results, for both load cases 1 and 2. It is concluded that they are in perfect agreement. As Table 1 indicates, failure occurs when $M$ equals 40 kNm/m.

Figure 4 illustrates the results of the load case 3. After elastic loading to 25 kNm/m and unloading to 0 kNm/m, plastic behaviour is observed during the reloading when $M$ exceeds 25 kNm/m (refer to Table 2). The second unloading remains elastic ($EI_1$), leading to residual (plastic) deformation equal to 0.054 mm. The final reloading curve follows the $M - \kappa$ diagram (Figure 2), but is shifted by the earlier developed plastic deformation.

In Figure 5 the results of the load case 4 are plotted. The difference between the previous load case is that instead of unloading to $M$ = 0 kNm/m, load is reversed and $M$ reaches -25 kNm/m. The first reverse loading is elastic but the second leads to a plastic component when $M$ exceeds -15 kNm/m. As specified in Table 1, the elastic loading/unloading branch applies for $M$ less than 25 kNm/m. After reloading to 35 kNm/m, elastic reverse loading occurs till -15 kNm/m (35-25-25 = -15 kNm/m). The resulting plastic deformation for $M$ = 0 kNm/m equals the one in load case 3.

In Figure 6 the results of the load case 5 are presented. In this case a hysteresis loop is studied. The beam is loaded close to failure (39.5 kNm/m) and afterwards, equal bending moment is applied in reverse direction. The unloading reveals accumulation of plastic deformation equal to 0.169 mm. The inclination of the reverse loading and reloading branches equals $EI_i$, with $i$ depending on the ratio $M/\kappa$.