PLAXIS LE CONSOLIDATION

Theory Manual

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1 INTRODUCTION

Consolidation problems in geotechnical and geo-environmental engineering involve the simultaneous solution of a series of partial differential equations (PDEs). The PDEs must be solved for all finite elements which when combined form a continuum representing the geometry of the problem. The theory of consolidation and its corresponding governing PDEs embrace the physical behavior of the geotechnical material and the laws of conservation of mass and momentum. Multiple constitutive models are required, including models for the flow of water and stress-strain relationships. These constitutive models are often nonlinear, with properties that are non-constant. For example, the hydraulic conductivity may be presented as a function of void ratio, the compressibility may change depending on the confining stress or on a yield criterium, and so on. In addition, the system of equations may consider a fixed or an updated frame of reference. In this case, a moving mesh is recommended, for example, for problems that present large displacements. The consideration of large displacements adds geometrical nonlinearity to the system of governing PDEs. Consolidation is a formidable problem with multiple nuances and complexities.

The purpose of the theory manual is to provide the user with details regarding the theoretical formulation of the PDEs as well as the numerical method used in the solution of such PDEs. The intent of the theory manual is not to provide an exhaustive summary of all theories associated with consolidation. Rather, the intent is to clearly describe details of the theory used in the PLAXIS LE software.

The basic theoretical foundation behind groundwater analyses is laid out in the Groundwater Theory Manual. This manual focuses on hydro-mechanical coupling.

Coupled water flow and stress-strain problems can be found in numerous situations. The classical consolidation analysis described in the Soil Mechanics literature is one example of a problem involving flow and equilibrium. Hydro-mechanical analyses can be employed in the prediction of settlements that occur over extended periods of time, pore-water pressure built up due to loading, and many type of combined pore-water pressure and load conditions.

PLAXIS LE is a numerical analysis software capable of solving consolidation models using the finite element method. PLAXIS LE embraces one-dimensional (1D), two-dimensional (2D) plane strain, and three-dimensional (3D) conditions. A user-friendly and streamlined interface allows the quick input of geometry, boundary conditions, and material properties. The software automatically generates the finite element mesh, saving a great deal of modeling time and allowing better control of the solution accuracy. Three-dimensional models that were considered extremely challenging to built in the past can now be created in a shorter time, thanks to the geometry input and automatic mesh generation system used by PLAXIS LE.

Consolidation models use the PLAXIS LE front-end interface for model setup, the PLAXIS LE - Consolidation finite-element engine for the numerical analysis, and the PLAXIS LE Output back-end interface for results visualization.

Consolidation is one module within the PLAXIS LE software. Other modules, such as PLAXIS Designer and SOILVISION SOILS found within PLAXIS LE may be used to generate input data.

This Theory Manual provides a concise review of the theory and formulations on which consolidation as implemented in PLAXIS LE is based. For details regarding the software operation and modeling guidelines, please consult the USER MANUAL and the CONSOLIDATION TUTORIAL MANUAL, included with the software.

2 BASIC EQUATIONS FOR SMALL STRAIN CONSOLIDATION (SSC)

The small strain consolidation theory supposes that a soil particle occupies the same location of the Euclidian space during the whole process of consolidation. It is also assumed that soil particles and pore water are incompressible. It will be greatly simplified to deduce the theory as described by Biot (1941).

2.1 STATIC ADMISSIBLE STRESS FIELD

Assuming a small amount of solid particles including pores is in equilibrium state, we have

$$\begin{cases} \dfrac{\partial \sigma'_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + \dfrac{\partial u_w}{\partial x} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma'_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + \dfrac{\partial u_w}{\partial y} = 0 \\ \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma'_z}{\partial z} + \dfrac{\partial u_w}{\partial z} = -\gamma \end{cases} \qquad [1]$$

where:

$\sigma'_x, \quad \sigma'_y, \quad \sigma'_z, \quad \tau_{xy}, \quad \tau_{yz}, \quad \tau_{zx}$ = effective stress components,

u_w = pore water pressure,

γ =unit weight of soil.

2.2 ADMISSIBLE STRAIN FIELD AND DISPLACEMENT FIELD

Displacement field $\{W\}^T = (W_x \quad W_y \quad W_z)$ represents displacement in x, y and z directions on a point (x, y, z). It can be expressed as below when strain is small.

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial W_x}{\partial x} \\ \dfrac{\partial W_y}{\partial y} \\ \dfrac{\partial W_z}{\partial z} \\ \dfrac{\partial W_x}{\partial y} + \dfrac{\partial W_y}{\partial x} \\ \dfrac{\partial W_y}{\partial z} + \dfrac{\partial W_z}{\partial y} \\ \dfrac{\partial W_z}{\partial x} + \dfrac{\partial W_x}{\partial z} \end{Bmatrix} \qquad [2]$$

where:

$\{\varepsilon\}^T = (\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx})$.

2.3 CONSTITUTIVE EQUATIONS

A constitutive relationship connects the stress and strain fields together. Effective stress field can be expressed in the following increment equation:

$$\{\Delta\sigma'\} = \{\sigma'\} - \{\sigma'_0\} \qquad [3]$$

In general, we can write the constitutive equations as follows:

$$\{\Delta\sigma'\} = [D]\{\Delta\varepsilon\} \qquad [4]$$

where:

$\{\Delta\sigma'\}$ = incremental effective stress vector,

$\{\Delta\varepsilon\}$ = incremental strain vector,

$[D]$ = Correlation matrix between incremental effective stress and strain.

2.4 SEEPAGE FIELD AND MASS CONSERVATIVE EQUATION

Based on Darcy's law, the velocity field of pore water flow in a porous media can be expressed as:

$$\{v\} = -[k]\{\nabla\}H \qquad [5]$$

where:

$\{v\}^T = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix}$ = velocity vector,

$[k] = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}$ = permeability matrix,

H = total head.

Based on the conservation condition of mass, the amount of pore water flow out of a small portion of soil is equal to the volumetric variation of it, that is:

$$\{\nabla\}^T\{v\} + \{a\}^T \frac{\partial}{\partial t}\{\varepsilon\} = 0 \qquad [6]$$

where:

$\{a\}^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$.

By substitution equation [5] into [6], we can have the mass conservation equation as:

$$\{\nabla\}^T\left([k]\{\nabla\}H\right) = \{a\}^T \frac{\partial}{\partial t}\{\varepsilon\} \qquad [7]$$

The total head used in Darcy's law can be expressed as:

$$H = u_w / \rho_w g + z \qquad [8]$$

where:

H = total head,

ρ_w = density of water,

g = acceleration of gravity,

z = elevation head.

Using the relationship between total head and pore water pressure for saturated soils, equation [7] can be expressed as:

$$\frac{1}{\rho_w g}\{\nabla\}^T\left([k]\{\nabla\}u_w\right)+\frac{\partial k_{xz}}{\partial x}+\frac{\partial k_{yz}}{\partial y}+\frac{\partial k_{zz}}{\partial z}=\{a\}^T\frac{\partial}{\partial t}\{\varepsilon\} \qquad \textbf{[9]}$$

Expanding equation [9], we get:

$$\frac{1}{\rho_w g}\left[\begin{array}{l}\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial u_e}{\partial x}+k_{xy}\frac{\partial u_e}{\partial y}+k_{xz}\frac{\partial u_e}{\partial z}\right)\\+\frac{\partial}{\partial y}\left(k_{xy}\frac{\partial u_e}{\partial x}+k_{yy}\frac{\partial u_e}{\partial y}+k_{yz}\frac{\partial u_e}{\partial z}\right)\\+\frac{\partial}{\partial z}\left(k_{xz}\frac{\partial u_e}{\partial x}+k_{yz}\frac{\partial u_e}{\partial y}+k_{zz}\frac{\partial u_e}{\partial z}\right)\end{array}\right]+\frac{\partial k_{xz}}{\partial x}+\frac{\partial k_{yz}}{\partial y}+\frac{\partial k_{zz}}{\partial z}=\frac{\partial \varepsilon_v}{\partial t} \qquad \textbf{[10]}$$

3 GOVERNING EQUATION OF LARGE-STRAIN CONSOLIDATION (LSC)

To solve consolidation problems considering large strains, the virtual work and continuity equations are derived considering large strains. The Jaumann rate of Cauchy stress is considered, along with an updated Lagrangian approach, as described in Bonet *et al.* (2008). This theory will be presented for the one-dimensional condition. The generalization to 2D and 3D conditions may be accomplished by following the principles presented in the small strain consolidation section. Note also that the sign conventions presented in the previous sections remain the same for the large strain theory.

3.1 CONCEPTUAL FRAMEWORK

Suppose a body at certain reference time $t = t_o$ occupies a certain region of the Euclidian space. The position of a material point at this time can be described by its position vector ***X***. Let the position vector of any material point be x at time t:

$$\mathbf{x} = \mathbf{x}(\boldsymbol{X}, t) \qquad [11]$$

the variable x describes the path of any material point, which, at $t = t_o$ is located at ***X***. The configuration at this time is called the reference configuration.

3.2 STRAIN AND STRAIN RATE

Suppose a material point ***X*** and another point ***X*** + *d**X*** in its neighborhood at some reference time t_0 changes to another configuration at time t, located at $\mathbf{x}(\boldsymbol{X}, t)$ and $\mathbf{x}(\boldsymbol{X} + d\boldsymbol{X}, t)$. In this case, we can define:

$$\mathrm{d}\mathbf{x} = \boldsymbol{F}\mathrm{d}\mathbf{X} \qquad [12]$$

where: ***F*** represents the relative deformation tensor in the neighborhood of the material point. This is called the deformation gradient tensor.

3.2.1 Deformation rate tensor and spin tensor

Let us consider two adjacent material points ***X*** and ***X*** + d***X*** at the reference configuration and located at ***x*** and ***x*** + *d**x*** at current time *t*. These two points are considered to have velocities of υ and $\upsilon + d\upsilon$. Then, the velocity gradient tensor ***L*** is defined by a linear transformation, as follows:

$$\mathrm{d}\boldsymbol{\upsilon} = \mathbf{L}\mathrm{d}\mathbf{x} \qquad [13]$$

The following relationship may be presented, between the velocity gradient tensor ***L*** and the deformation gradient tensor ***F***:

$$\mathbf{L} = \mathbf{F} \cdot \mathbf{F}^{-1} \qquad [14]$$

The general velocity gradient tensor ***L***, may be decomposed into the sum of a symmetric and anti-symmetric part, as follows:

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \qquad [15]$$

where:

D	= [**L**]s [the symmetric part] is known as the rate of deformation or the stretching tensor, and
W	= [**L**]a [the anti-symmetric part] is known as the spin tensor

3.2.2 Cauchy stress and 1st Piola-Kirchholf stress

The Cauchy stress $\boldsymbol{\sigma}$ is defined with respect to the current configuration. The 1st Piola-Kirchholf stress ***S***, also known as the nominal stress, is defined on the reference configuration. The nominal stress ***S*** is commonly used to depict the motion and equilibrium of a body, for its simplicity. The relation between $\boldsymbol{\sigma}$ and ***S*** is as follows:

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \qquad [16]$$

where:

J	= det(**F**) **–** is the Jacobian between the current and reference configuration

Let us consider an arbitrary volume v in space. If a continuous medium of density ρ fills the volume at time t, the total mass in volume v is defined by

$$\mathrm{m} = \int_{v} \rho dv \qquad [\ 17\ ]$$

If no mass is added of removed from v, the mass m must independent of time t, i.e.

$$\dot{\mathrm{m}} = 0 \qquad [\ 18\ ]$$

We may transform the integral of Equation [17] to the reference configuration by using *dv = JdV*, resulting in:

$$\mathrm{m} = \int_{V} \rho J dV \qquad [\ 19\ ]$$

By taking the derivative, the following equation is obtained:

$$\int_{v} \big(\dot{\rho} + \rho \mathrm{div}(\mathbf{v})\big) dv = 0 \qquad [\ 20\ ]$$

Extending the same concept to any part of the body, the continuity equation is obtained as:

$$\dot{\rho} + \rho \mathrm{div}(\mathbf{v}) = 0 \qquad [\ 21\ ]$$

3.2.3 Momentum principle - equations of motion and equilibrium

The Momentum principle for a collection of material points states that the time rate of change of the total momentum of a given set of material points equals the vector sum of all the external forces acting on the material points of the set, assuming that Newton's Third Law governs the internal forces. Based on the Momentum principle, the following equation may be presented:

$$\mathrm{div}(\boldsymbol{\sigma}) + \rho \boldsymbol{g} = \rho \boldsymbol{a} \qquad [\ 22\ ]$$

These are the equations that must be satisfied for any continuum, whether it is a solid or a fluid in motion. These equations are also known as the Cauchy's equations of motion. If the acceleration vanishes, the equilibrium equations are reduced to the static condition, as follows:

$$\mathrm{div}(\boldsymbol{\sigma}) + \rho \boldsymbol{g} = \mathbf{0} \qquad [\ 23\ ]$$

Similarly, based on the nominal stress (in the reference configuration at t_o), the equilibrium equations become:

$$\mathrm{div}\big(\boldsymbol{S}_{t_o}\big) + \rho_o \boldsymbol{g} = \mathbf{0} \qquad [\ 24\ ]$$

where:

ρ_o = denotes the density on the reference configuration at time t_0

$\boldsymbol{S}_{t_o}$ = the nominal stress tensor in the reference configuration at t_0

Taking the material time derivatives for Equation [24], the following equation is obtained:

$$\mathrm{div}\big(\dot{\boldsymbol{S}}_{t_o}\big) + \rho_o \dot{\boldsymbol{g}} = \mathbf{0} \qquad [\ 25\ ]$$

In most geotechnical engineering problems, the rate of gravity acceleration is negligible. Then, equation [25] reduces to

$$\mathrm{div}\big(\dot{\boldsymbol{S}}_{t_o}\big) = \mathbf{0} \qquad [\ 26\ ]$$

Taking the current configuration as the reference configuration, the equilibrium equations based on the nominal stress rate become:

$$\mathrm{div}\big(\dot{\boldsymbol{S}}_{t}\big) = \mathbf{0} \qquad [\ 27\ ]$$

where:

$\dot{\boldsymbol{S}}_t$ = the rate of nominal stress which takes the current configuration at time t as the reference configuration

3.2.4 The moment of momentum principle

In a collection of material points whose interactions are equal, opposite, and collinear forces, the time rate of change of the total moment of momentum for the given collection of material points is equal to the vector sum of the moments of the external forces acting on the system. Based on the moment of momentum principle, the Cauchy stress becomes a symmetric tensor:

$$\sigma_{ij} = \sigma_{ji} \tag{28}$$

which is written in a component form.

3.2.5 Principle of virtual work

Suppose that a body is in a certain equilibrium configuration, and that all the material points of the body are given an infinitesimal virtual strain rate tensor $\delta\upsilon$ from the equilibrium configuration. The virtual strain rate tensor is a function of the position within the body, and has continuous first-order partial derivatives with respect to x. It is common to prescribe υ along certain boundary surfaces and in this case, $\delta\upsilon$ = **0**.

Applying the divergence theorem, the principle of virtual work [power] in terms of the first Piola-Kirchholf stress tensor on the reference configuration can be expressed as:

$$\int_v (\dot{\boldsymbol{S}}_t \cdot \delta \boldsymbol{L}) dv = \int_\Gamma (\dot{\boldsymbol{s}}_t \cdot \delta \mathbf{v}) d\Gamma \tag{29}$$

where:

$\dot{\boldsymbol{S}}_t$	= is the nominal stress rate
$\dot{\boldsymbol{s}}_t$	= is the nominal load rate along the domain boundary

3.2.6 Conservation equation of mass of moisture

Applying the principle of conservation of mass of water stored in the soil, the following equation is obtained:

$$\frac{\partial}{\partial y}\left[k_y \frac{\partial \phi}{\partial y}\right] = \frac{\partial \varepsilon_v}{\partial t} \tag{30}$$

where:

k_y	= hydraulic conductivity for unsaturated soil, m/s,
ϕ	= total water head, m,
ε_v	= volume strain, and $\varepsilon_v = \varepsilon_y$ in 1D case.

3.2.7 Constitutive equations

The equations derived above, which are applicable to any continuous medium, will not be sufficient in number to determine all the unknowns. Constitutive equations characterizing the material, are required.

The current stress-strain constitutive models available in PLAXIS LE for large strain consolidation analysis only consider saturated conditions. According to Bishop, the effective stress for saturated soils was first expressed as:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \gamma_w \phi \boldsymbol{I} \tag{31}$$

where:

$\boldsymbol{I}$	= identity tensor

According to the principle of material frame indifference or principle of material objectivity, it is required that the stress rate used in the constitutive equations should be indifferent with the reference frame. It has been verified by various researchers that the Jaumann rate of Cauchy stress is a suitable stress rate, with objectivity. The Jaumann rate related to intrinsic rate of Cauchy stress is given by (Bonet *et al.*, 2008):

$$\mathring{\boldsymbol{\sigma}}' = \dot{\boldsymbol{\sigma}}' - \boldsymbol{W} \cdot \boldsymbol{\sigma}' + \boldsymbol{\sigma}' \tag{32}$$

The constitutive equation for saturated soils is presented assuming that the soil behaves as an isotropic, linear or non-linear elastic material. The soil structure constitutive relationship can be written in the following rate form

$$\mathring{\boldsymbol{\sigma}}' = \Theta \dot{\varepsilon} \tag{33}$$

where:

$\mathring{\boldsymbol{\sigma}}'$	= the Jaumann rate of Cauchy's stress,

Θ = Correlation tensor between stress rate and strain rate,

$\dot{\varepsilon}$ = strain rate.

In a one-dimensional condition, two equations govern the consolidation phenomenon, one is the equilibrium equation and the other is the water mass continuity equation. Two- and three-dimensional problems include additional equations for the equilibrium in the added dimensions. A fully coupled analysis is required to correctly model soil displacement and the pore-water pressure response to an applied load. Solving all these equations simultaneously by using the updated Lagrangian method is done using as main variables the displacements and pore-water pressures.

The constitutive models available to represent the stress-strain relationship and to represent hydraulic conductivity are presented in the USER MANUAL. These relationships may be used in small-strain analyses. Some models are exclusively developed for large strain simulations because they were conceptualized to represent soft materials, such as tailings.

4 REFERENCES

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