A Finite Element Formulation for a Beam with Varying Cross-Section Geometry along Its Length: Flexibility-Based Approach

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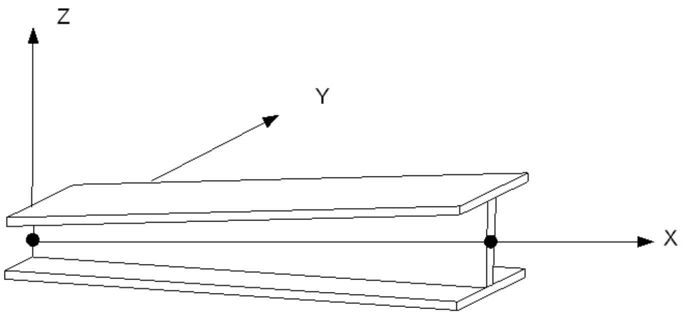
Introduction

This study addresses a finite element formulation for a beam element with changing cross-section geometry along its length. The study utilizes a flexibility-based formulation. In this method, internal element forces (axial and bending) are used to derive exact form of element stiffness matrix. To this end, equilibrium equations are fully satisfied along the element, which makes it sufficient to use one element per member to capture accurate results. This is in contrast with conventional displacement-based formulation in which displacements fields are used to obtain element stiffness matrix and equailibrium equations are only satisfied in a weighted integral form. Therefore, one needs to use more than one element per member to capture accurate results.

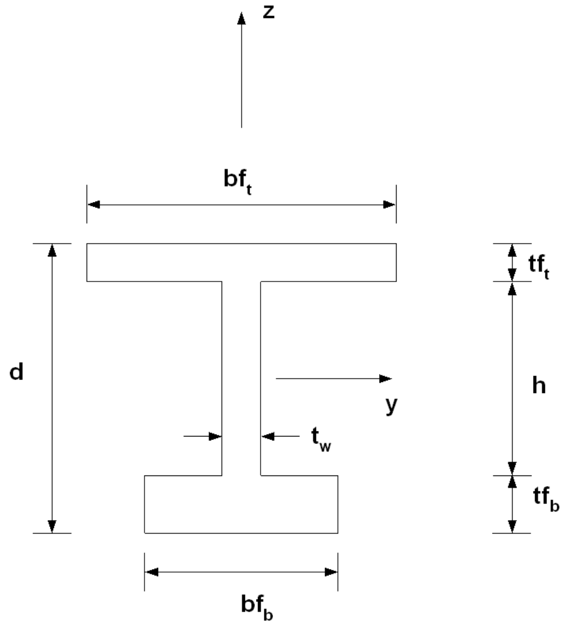
In particular, a web-tapered beam element is chosen as an example but the formulation steps given in the study can be easily extended to beams with different cross-section variation. Due to varying cross-section geometric properties along the length, the centroid axis is not a straight line, but rather a curved line. Hence, the proposed element considers changes in the centroid axis and calculate the stiffness matrix that includes effects of the curved centroid axis.

An elastic material behavior is assumed and no geometric nonlinear effects are considered. In addition, it is assumed that deformations are small and section remains section after the deformation. This study also provides a method to include shear deformations within the proposed element. In addition, it is shown how one can include distributed load effects on the element response. Finally, two numerical examples are provided to show the merits of the proposed element.



Geometry Definition on Cross-section and Along Beam's Length

The element cross-section definition is given in the following figure in which unequal flange lengths and thicknesses are used. Also, a linear variation of element depth ("h" in the figure) is assumed in this study.



Cross-section Area, A(x)

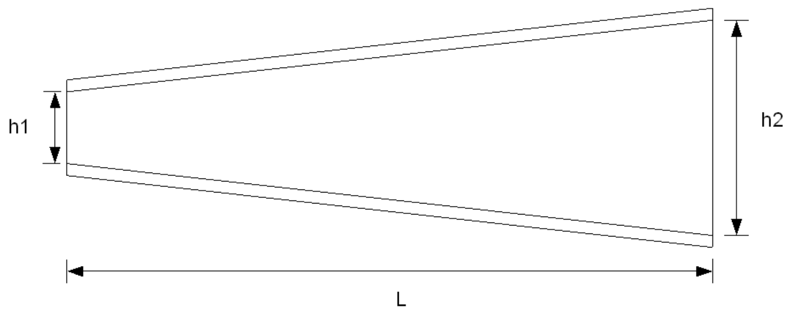
Referring to the figure above, the cross-section area at any point along the element can be represented as follows:

$$A(x) = tf_t\, bf_t + tf_b\, bf_b + h(x)\, t_w \tag{1}$$

in which "x" is the web height of the member. A linear variation of "h" is assumed:

$$h(x) = \frac{h2 - h1}{L} x + h1 \tag{2}$$

where h1 and h2 are shown in the following figure.



Combining Eq. (1) and (2), it is obtained that

$$A(x) = A1 + \alpha 1\, x \tag{3}$$

in which

$$A1 = tf_t\, bf_t + tf_b\, bf_b + h1\, t_w$$
$$\alpha 1 = \frac{(h2 - h1)}{L}\, t_w \tag{4}$$

and A1 is the area of the cross-section at left end of the element. It should be noted that A(x)=A1 if $\alpha 1 \to 0$. In other words, it is a constant cross-section (prismatic element) if $\alpha 1 \to 0$.

■ Cross-section Torsion Constant, J(x)

Section torsional constant is expressed according to the following equation:

$$J(x) = \frac{1}{3}\, tf_t^3\, bf_t + \frac{1}{3}\, tf_b^3\, bf_b + \frac{1}{3}\, t_w^3\, h(x) \tag{5}$$

where it is assumed that the cross-section is composed of thin components. Thus,

$$J(x) = J1 + \beta 1\, x \tag{6}$$

where

$$J1 = \frac{1}{3}\, tf_t^3\, bf_t + \frac{1}{3}\, tf_b^3\, bf_b + \frac{1}{3}\, t_w^3\, h_1$$
$$\beta 1 = \frac{(h2 - h1)}{L}\, \frac{1}{3}\, t_w^3 \tag{7}$$

Again, note that J1 is the section torsion constant at left end of the element.

■ Cross-section Moment of Inertia: I_{yy}(x)

In order to calculate I_{yy}, position of cross-section centroid is needed. Note that the centroid axis is a curved line due to effects of varrying cross-section geometry along the element. This can be explicitly expressed as follows:

$$\hat{z}(x) = \left(\frac{1}{2}\, b_{fb}\, t_{fb}^2 + b_{ft}\left(h(x) + t_{fb} + \frac{t_{ft}}{2}\right) t_{ft} + h(x)\left(\frac{h(x)}{2} + t_{fb}\right) t_w\right) \Big/ (b_{fb}\, t_{fb} + b_{ft}\, t_{ft} + h(x)\, t_w) \tag{8}$$

where "$\hat{z}$ (x)" is measured from the bottom of the cross-section (see the figure). Finally, an explicit form of I_{yy} is obtained as follows:

$$I_{yy}(x) = \frac{1}{12} b_{ft} t_{ft}^3 + b_{ft} t_{ft} \left(h(x) + t_{fb} + \frac{t_{ft}}{2} - z\hat{(x)}\right)^2 + \frac{1}{12} b_{fb} t_{fb}^3 + b_{fb} t_{fb} \left(\hat{z}(x) - \frac{t_{fb}}{2}\right)^2 + \frac{1}{12} t_w (h(x))^3 + t_w h(x) \left(\frac{h(x)}{2} + t_{fb} - \hat{z}(x)\right)^2; \quad (9)$$

- **Cross-section Moment of Inertia: $I_{zz}(x)$**

Since the cross-section has a symmetry along z-z axis, the position of the centraid is always at y = 0. Thus, the following immediately applies:

$$I_{zz}(x) = \frac{1}{12} b_{ft}^3 t_{ft} + \frac{1}{12} b_{fb}^3 t_{fb} + \frac{1}{12} t_w^3 h(x); \quad (10)$$

Element Formulation: Flexibility Based Approach

The principle of virtual work can be expressed as follows:

$$\int_0^{L_o} \delta\mathbf{F}^T \mathbf{d} \, dx - \delta\mathbf{Q}^T \mathbf{q} = 0 \quad (11)$$

where **F** is the weigted function (which is chosen in such a way that it satisfies equilibrium equations), **d** is the generalized cross-section strains, **Q** is the force vector applied at element ends and finally, **q** is the displacement vector. It should be noted that the above integration is carried out at initial position (i.e., L_o is the initial length of the element).

A consistent linearization of Eq. (11) leads to element flexibility matrix. Details of the linearization steps are not given in this study but the reader is referred to the following publication: ***Alemdar, B.N. and White, D.W., (2005), Displacement, Flexibility and Mixed Beam-Column Finite Element Formulation for Distributed Plasticity Analysis, Journal of Structural Engineering, Vol. 131, No. 12, pg. 1811-1815***.

In general form, the flexibility matrix for axial, torsional and bending terms are respectively expressed as follows:

$$\mathbf{F_A} = \int_0^{L_o} \frac{1}{E A(x)} (\mathbf{D_A}(x))^T \mathbf{D_A}(x) \, dx \quad (12)$$

$$\mathbf{F_J} = \int_0^{L_o} \frac{1}{G J(x)} (\mathbf{D_J}(x))^T \mathbf{D_J}(x) \, dx \quad (13)$$

$$\mathbf{F_{Byy}} = \int_0^{L_o} \frac{1}{E I_{yy}(x)} (\mathbf{D_{yy}}(x))^T \mathbf{D_{yy}}(x) \, dx \quad (14)$$

$$\mathbf{F_{Bzz}} = \int_0^{L_o} \frac{1}{E I_{zz}(x)} (\mathbf{D_{zz}}(x))^T \mathbf{D_{zz}}(x) \, dx \quad (15)$$

in which E and G are modulus of elasticity and shear, respectively, and A(x), J(x), $I_{yy}(x)$ and $I_{zz}(x)$ are cross-section geometry functions which are derived in the previous sections for a web-tapered beam element. Finally, matrices **D**(x) are referred to as interpolated stress-resultant force fields and they satisfy equilibrium equations. This is further explained in coming sections.

It should be noted that Eqs. (12) - (15) are exact flexibility matrices of the proposed element. Exact stiffness matrices are simply the inverse of the flexibility matrices. The inverse of the flxibility matrices always exists since **D**(x) contains the section forces corresponds to deformations only (i.e. **D**(x) does not contain any rigid body modes).

Flexibility Matrix for Axial Deformations

In the absence of element loading, the axial force inside the element is constant. Hence, interpolated internal axial force can be expressed as follows:

$$D(x) = \mathbf{D_A}^T \mathbf{Q} = (1)\,(Q_1) \tag{16}$$

in which Q_1 is the axial force in the element. Then, Eq. (12) becomes

$$\mathbf{F_A} = \int_0^{L_o} \frac{1}{E\,A(x)}\,dx = \frac{\text{Log}\left[1+\frac{L_o\,\alpha 1}{A1}\right]}{E\,\alpha 1} \tag{17}$$

It is interesting to note that

$$\text{Limit}_{\alpha 1\to 0}\ \frac{\text{Log}\left[1+\frac{L_o\,\alpha 1}{A1}\right]}{E\,\alpha 1} = \frac{L_o}{E\,A1} \tag{18}$$

In other words, the above term reduces to conventional flexibility term for a prismatic element.

Flexibility Matrix for Torsinal Deformations

Torsinal stiffness term is obtained in a similar way that the axial term is obatined. Again, under no torsional loading, one can write that

$$D(x) = \mathbf{D_J}^T \mathbf{Q} = (1)\,(Q_2) \tag{19}$$

in which Q_2 is the constant torsional force in the element. Then, Eq. (13) becomes

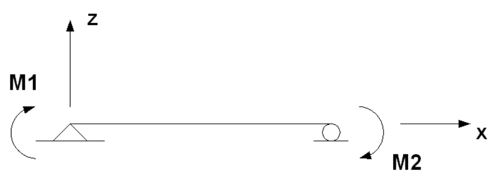
$$\mathbf{F_J} = \int_0^{L_o} \frac{1}{GJ(x)}\,dx = \frac{\text{Log}\left[1+\frac{L_o\,\beta 1}{J1}\right]}{G\,\beta 1} \tag{20}$$

It is again interesting to note that

$$\text{Limit}_{\beta 1\to 0}\ \frac{\text{Log}\left[1+\frac{L_o\,\beta 1}{J1}\right]}{G\,\beta 1} = \frac{L_o}{G\,J1} \tag{21}$$

Element Formulation for Bending in Major Axis

If the element is subjected to end moments only, the moment field inside the element is linear.



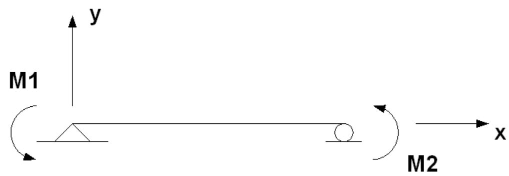
Thus,

$$D(x) = \mathbf{D_{yy}}^T \mathbf{Q} = \begin{pmatrix} 1-\frac{x}{L} & -\frac{x}{L} \end{pmatrix} \begin{pmatrix} M1 \\ M2 \end{pmatrix} \tag{22}$$

One can substitute $\mathbf{D_{yy}}$ into Eq. (14) to obtain the flexibility terms for major axis bending. Due to higly nonlinear nature of I_{yy}, it is not practical to obtain a closed form solution. Instead, numerical evaluate of Eq. (14) is preferred.

Element Formulation for Bending in Minor Axis

The stiffness terms for bending in minor axis follows the similar steps given in the previos section. Referring to the following figure, one can write



$$D(x) = \mathbf{D_{zz}}^{\mathbf{T}}\mathbf{Q} = \begin{pmatrix} \frac{x}{L} - 1 & \frac{x}{L} \end{pmatrix} \begin{pmatrix} M1 \\ M2 \end{pmatrix} \tag{23}$$

And then, Eq. (15) is again numerically evaluated.

■ Shear Deformations Due to Bending

Shear deformations due to bending are integrated into the formulation in such a way that an equivalent shear area concept is utilized. The complementary internal virtual work due to shear deformations is

$$\delta W_{int}^{*} = \int_{V_o} \delta\sigma \frac{1}{G} \sigma \, dV \tag{24}$$

where $\sigma = \tau = \frac{V}{A_s}$, V: shear at cross-section and $A_s$is equivalent shear area of the cross-section. It should be noted that the above integration is carried out over the initial volume of the element. Then, substituting $dV = A_s \, dx$, Eq. (24) becomes

$$\delta W_{int}^{*} = \int_{V_o} \left(\frac{\delta V}{A_s}\right) \frac{1}{G} V \, dx \tag{25}$$

Shear stresses due to applied end moments are

$$V = \begin{pmatrix} \frac{1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{L} & \frac{1}{L} \end{pmatrix} \mathbf{M} \tag{26}$$

and equivalenet shear area along the member is assumed to be changing linearly:

$$A_s(x) = A_{s1}\left(1 - \frac{x}{L}\right) + A_{s2}\left(\frac{x}{L}\right) \tag{27}$$

where A_{s1} and A_{s2} are the equivalent shear areas at element left and right end, respectively. Thus, Eq. (26) becomes

$$\delta W_{int}^{*} = \delta \mathbf{M} \left(\frac{1}{G} \frac{1}{L^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \int_0^L \frac{1}{A_s(x)} dx \right) \mathbf{M} \tag{28}$$

or, flexibility matrix due to shear deformations under major axis bending is

$$Fs_{maj} = \frac{1}{G} \frac{1}{L^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \int_0^L \frac{1}{A_s(x)} dx \tag{29}$$

Note that this matrix should be added to Eq. (14) before taking the inverse of the flexibility matrix. Again, numerical integration can be used to evaluate the integral.

A similar approach is also followed for shear deformations due to bending in minor axis. Thus,

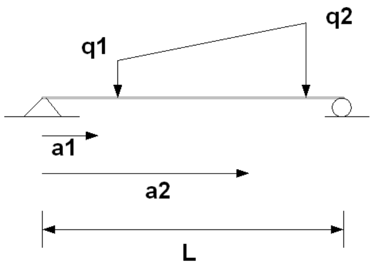
$$Fs_{min} = \frac{1}{G} \frac{1}{L^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \int_0^L \frac{1}{A_s(x)} dx \tag{30}$$

where $A_s(x)$ represents equivalent shear areas under minor axis bending.

■ Distributed Load (Major Axis Loading)

Distributed loads effect on the element response can be also included within the flexibility-based formulation. In this section, a trapezoidal load is studied as an example to show how to calculate fixed end moment effects. Other types of loads can be also handled in a similar way.

The element is subjected to a trapezoidal load as shown below and the load definition F(x) is



$$F(x) = \left(\frac{q1 - q2}{a1 - a2}\right) x + \left(\frac{a1\, q2 - a2\, q1}{a1 - a2}\right) = \beta_1 x + \beta_2 \tag{31}$$

And one can calculate corresponding deflections due to the applied load as follows:

$$\boldsymbol{\theta} = \begin{pmatrix} \theta 1 \\ \theta 2 \end{pmatrix} = \int_0^{L_0} \frac{\mathbf{D_{zz}}^T M(x)}{E\, I_{yy}(x)}\, dx \tag{32}$$

in which $\mathbf{D_{zz}}$ and $I_{yy}(x)$ are explicity derived in previous sections. Note that M(x) is the moment field due to applied load F(x) and it is given below:

$$M(x) = \begin{cases} V_1 x & 0 \le x < a1 \\ V_1 x - q1 \frac{(x-a1)^2}{2} - (F(x) - q1) \frac{(x-a1)^2}{6} & a1 \leq x \leq a2 \\ V_1 x - q1 (a2 - a1) \left(\frac{a2-a1}{2} + x - a2\right) - (q2 - q1) \left(\frac{a2-a1}{2}\right) \left(\frac{a2-a1}{3} + x - a2\right) & a2 \leq x \leq L \end{cases} \tag{33}$$

where

$$V_1 = \frac{1}{L} \left(q1 (a2 - a1) \left(\frac{a2 - a1}{2} + L - a2\right) + (q2 - q1) \left(\frac{a2 - a1}{2}\right) \left(\frac{a2 - a1}{3} + L - a2\right)\right) \tag{34}$$

And finally, one can convert calculated end rotations to equivalent "Fixed End Moments" as follows

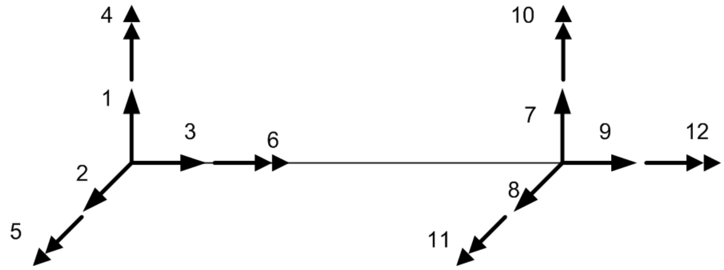
$$\mathbf{F_f} = \begin{pmatrix} Mf_1 \\ Mf_2 \end{pmatrix} = \mathbf{K_{Byy}}\, \boldsymbol{\theta} \tag{35}$$

in which $\mathbf{K_{Byy}}$ is the element stiffness matrix, which is the inverse of the flexibility matrix given in Eq. (14).

Closing Remarks

The stiffness matrix developed in the previous sections must be augmented with rigid body modes to obtain a complete form of element stiffness matrix. The stiffness matrix given in the previous section is deu to the deformations modes, in which rigid body modes are excluded (see the first figure below). Then, one needs to add rigid body modes (hence, size of the stiffness matrix is chaged from (6,6) to (12,12)) and this can be accomplished with a transformation matrix T.

$$\mathbf{K} = \mathbf{T}^T \mathbf{K_e}\, \mathbf{T} \tag{36}$$



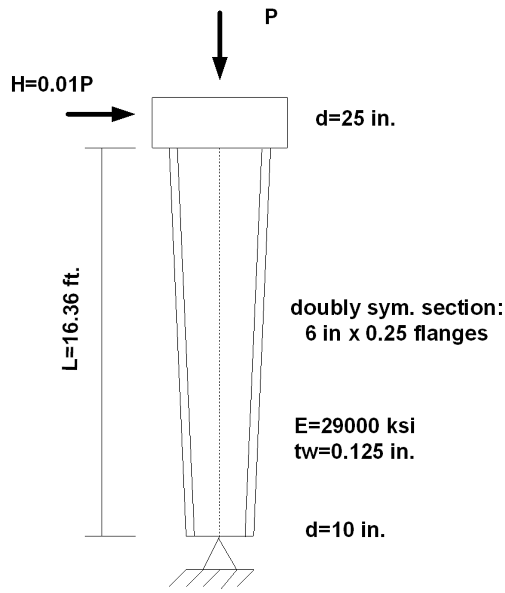
Numerical Examples:

Two numerical examples are provided in this study, which are also used as a benchmark test examples in "***AISC Steel Design Guide 25: Frame Design Using Web-Tapered Members***". In both examples, the element stiffness matrix is calculated based on the proposed element formulation and compared to those given in the reference. Note that only one element per member is sufficient because the exact form of stiffness terms are calculated.

- **Doubly Symmetric Web-Tapered Beam-Column:**

The problem definition is provided in the figure below. The following numerical values are used:

$$\mathrm{bf}_t = 6 \text{ in.} \quad \mathrm{tf}_t = 0.25 \text{ in.} \quad \mathrm{bf}_b = 6 \text{ in.} \quad \mathrm{tf}_b = 0.25 \text{ in.} \quad t_w = 0.125 \text{ in.}$$
$$\mathrm{h2} = 25 - 2 * 0.25 = 24.5 \text{ in.} \quad \mathrm{h1} = 10 - 2 * 0.25 = 9.5 \text{ in.}$$
$$E = 29\,000 \text{ ksi} \quad G = 11\,153.85 \text{ ksi}$$
$$L = 16.36 * 12 = 196.32 \text{ in.}$$



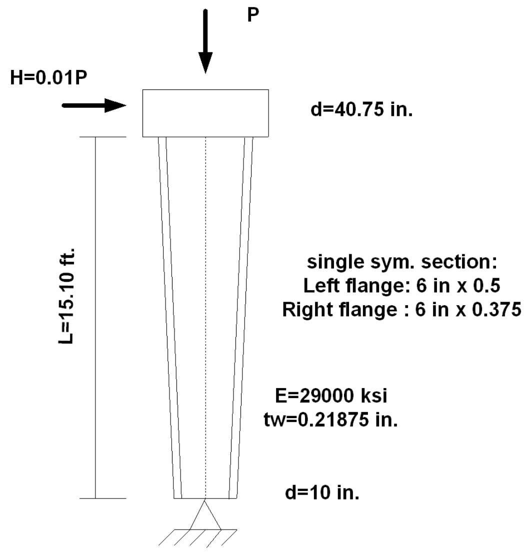
$$K = \begin{pmatrix} 11.511 & 0 & 0 & 0 & -760.578 & 0 & -11.511 & 0 & 0 & 0 & -1499.26 & 0 \\ 0 & 0.414058 & 0 & 40.6421 & 0 & 0 & 0 & -0.414058 & 0 & 40.6458 & 0 & 0 \\ 0 & 0 & 748.534 & 0 & 0 & 0 & 0 & 0 & -748.534 & 0 & 0 & 0 \\ 0 & 40.6421 & 0 & 5319.12 & 0 & 0 & 0 & -40.6421 & 0 & 2659.74 & 0 & 0 \\ -760.578 & 0 & 0 & 0 & 82213.7 & 0 & 760.578 & 0 & 0 & 0 & 67103.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.17358 & 0 & 0 & 0 & 0 & 0 & -4.17358 \\ -11.511 & 0 & 0 & 0 & 760.578 & 0 & 11.511 & 0 & 0 & 0 & 1499.26 & 0 \\ 0 & -0.414058 & 0 & -40.6421 & 0 & 0 & 0 & 0.414058 & 0 & -40.6458 & 0 & 0 \\ 0 & 0 & -748.534 & 0 & 0 & 0 & 0 & 0 & 748.534 & 0 & 0 & 0 \\ 0 & 40.6458 & 0 & 2659.74 & 0 & 0 & 0 & -40.6458 & 0 & 5319.84 & 0 & 0 \\ -1499.26 & 0 & 0 & 0 & 67103.1 & 0 & 1499.26 & 0 & 0 & 0 & 227232. & 0 \\ 0 & 0 & 0 & 0 & 0 & -4.17358 & 0 & 0 & 0 & 0 & 0 & 4.17358 \end{pmatrix}$$

- **Doubly Symmetric Web-Tapered Beam-Column:**

The problem definition is provided in the figure below. The following numerical values are used:

$$\text{bf}_t = 6\ \text{in.}\quad \text{tf}_t = 0.5\,\text{in.}\quad \text{bf}_b = 6\,\text{in.}\quad \text{tf}_b = 0.375\ \text{in.}\quad t_w = 0.21875\,\text{in.}$$
$$\text{h2} = 40.75 - 2 * 0.4375 = 39.875\ \text{in.}\quad \text{h1} = 10 - 2 * 0.4375 = 9.125\ \text{in.}$$
$$E = 29\,000\ \text{ksi}\quad G = 11\,153.85\,\text{ksi}$$
$$L = 15.10 * 12 = 181.2\,\text{in.}$$

The second example is given below. Member cross section is defined with unequal flange lengths. The following numerical values are used for the example:



$$K = \begin{pmatrix} 51.4257 & 0 & 0 & 0 & -2374.56 & 0 & -51.4257 & 0 & 0 & 0 & -6943.79 & 0 \\ 0 & 0.922516 & 0 & 83.5563 & 0 & 0 & 0 & -0.922516 & 0 & 83.6037 & 0 & 0 \\ 0 & 0 & 1639.48 & 0 & 0 & 0 & 0 & 0 & -1639.48 & 0 & 0 & 0 \\ 0 & 83.5563 & 0 & 10092.2 & 0 & 0 & 0 & -83.5563 & 0 & 5048.23 & 0 & 0 \\ -2374.56 & 0 & 0 & 0 & 207819. & 0 & 2374.56 & 0 & 0 & 0 & 222450. & 0 \\ 0 & 0 & 0 & 0 & 0 & 27.0087 & 0 & 0 & 0 & 0 & 0 & 27.0087 \\ -51.4257 & 0 & 0 & 0 & 2374.56 & 0 & 51.4257 & 0 & 0 & 0 & 6943.79 & 0 \\ 0 & -0.922516 & 0 & -83.5563 & 0 & 0 & 0 & 0.922516 & 0 & -83.6037 & 0 & 0 \\ 0 & 0 & -1639.48 & 0 & 0 & 0 & 0 & 0 & 1639.48 & 0 & 0 & 0 \\ 0 & 83.6037 & 0 & 5048.23 & 0 & 0 & 0 & -83.6037 & 0 & 10100.8 & 0 & 0 \\ -6943.79 & 0 & 0 & 0 & 222450. & 0 & 6943.79 & 0 & 0 & 0 & 1.03576 \times 10^{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 27.0087 & 0 & 0 & 0 & 0 & 0 & 27.0087 \end{pmatrix}$$

Conculusion

In this study, a flexibility-based element formulation is adapted to derive exact stiffness matrix for a beam with varying geometric cross-section properties along its length. The proposed element considers curved centroid axis position within its stiffness formulation. Calculated stiffness terms are exact (the only approximation introduced into the formulation comes from numerical integration of corresponding stiffness terms.). To this end, one element per member is sufficient to capture accurate results for members with varying cross-sections. Shear deformations and effects of distributed loads on the element are also studied in this study.

The proposed formulation particularly targets beams with linearly tapered web, but the concept presented in this paper can be easily extended to other type of varying cross-sections. Currently, the proposed element is used in a structural software poduct , **RAM Elements** (www.bentley.com/en-us/products/ram%20elements/).

Author

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